On the Practical Computation of Stokes Matrices

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Joint work (in progress) with Michèle Loday-Richaud and Pascal Rémy

Main Result

Setting: $L = a_r(x)\,\frac{d^r}{dx^r} + \cdots + a_1(x)\,\frac{d}{dx} + a_0,$ $\qquad a_i \in \mathbb{Q}[x]$

$a_r(0) = 0$ (singular point)

Task. Compute the **Stokes matrices** of L at 0.

Key features.

(a) implemented

(b) fully automatic

(c) for arbitrary L of **pure level 1**

(d) no numeric Laplace transforms

(e) error bounds

Related Work

	(a)	(b)	(c)	(d)	(e)
• Mathematical theory [Horn, Trjitzinski, Turrittin, …, Ramis, Écalle, ∼1910–1990]			✓$^+$	✓	✓
• Thomann, Fauvet, Richard-Jung (∼1990–2010)	✓	?		✓	
• van der Hoeven (2007)		✓	✓$^+$		✓
• Loday-Richaud, Rémy (2012)			∼	✓	

(a) implemented

(b) fully automatic

(c) for arbitrary L of **pure level 1**

(d) no numeric Laplace transforms

(e) error bounds

Motivation: Generators of the Differential Galois Group

$$L = a_r(x)\,\frac{d^r}{dx^r} + \cdots + a_1(x)\,\frac{d}{dx} + a_0$$

Theorem (Ramis, 1985)**.** The differential Galois group of L is the algebraic group generated by:

- the monodromy matrices,
- the exponential torus,
- the Stokes matrices

(all viewed as elements of $GL_r(\mathbb{C})$ acting on local solutions at a base point x_0).

Application (van de Hoeven, 2007)**.**
Symbolic-numeric algorithms for exact solutions of differential equations.

[also Llorente (2014), Chyzak–Goyer–M. (2022)]

Formal Solutions

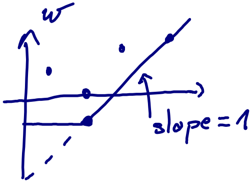
Theorem (Fabry, 1885)**.** The operator L has a full basis of **formal solutions** of the form

$$e^{q(1/x^p)}\, x^{\lambda} \sum_{k=0}^{r} \sum_{n=0}^{\infty} c_{k,n}\, x^{n/p} \log(x)^k,$$

- $q(1/x^p) \in \bar{\mathbb{Q}}[x^{1/p}]$
- $\lambda \in \bar{\mathbb{Q}}$
- $f_k(x^{1/p}) = \sum_n c_{k,n}\, x^{n/p} \in \bar{\mathbb{Q}}[[x^{1/p}]]$, usually divergent (cvgce rad. $= 0$)

Assumption. The origin is a singular point of **pure level 1**, i.e., the exponential parts are all of the form $e^{\alpha/x}$.

- This implies $p = 1$ (no ramification anywhere).
- No exp parts $\Leftrightarrow$ regular singular $\Rightarrow$ the f_k are convergent



$$w\left(x^j \frac{d^i}{dx^i}\right) = j - i$$

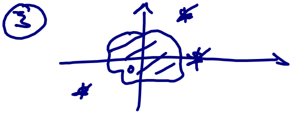
Borel Summation

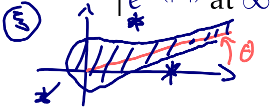
"Definition". A series $y(x) = x^{\lambda}\,\mathbb{C}[[x]][\log x]$ with $\mathrm{Re}(\lambda) > 0$ is **Borel-summable** in the direction θ when the following steps all make sense:

$$y(x) \in x^{\lambda}\,\mathbb{C}[[x]][\log x] \qquad\qquad \mathcal{S}_{\theta}(y)(x) = \int_0^{e^{i\theta}\infty} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$

Borel: $\mathcal{B}(x^{\nu}\log(x)^{k}) = \frac{d^k}{d\nu^k}\frac{\xi^{\nu-1}}{\Gamma(\nu)}$ $\downarrow$ $\qquad\qquad$ $\uparrow$ Laplace

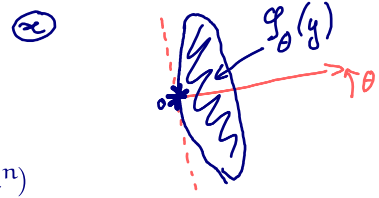
$\hat{y}(\xi)$ convergent $\quad \xrightarrow{\text{analytic continuation}} \quad$ $\hat{y}(\xi)$ | analytic on $e^{i\theta}[0,\infty)$, $e^{O(|x|)}$ at ∞





- Solutions of L are Borel-summable
- The resulting $\mathcal{S}_{\theta}(y)$ is a **solution** of L **asymptotic** to y on a domain of opening π:

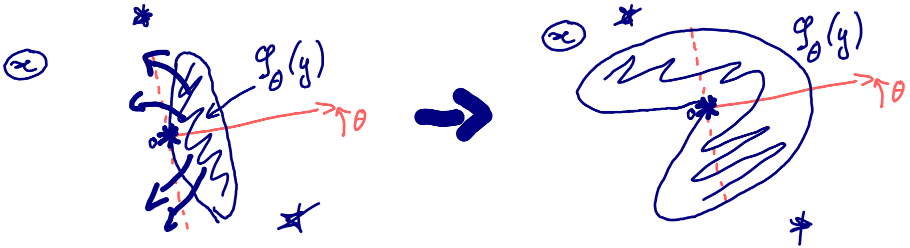
$$\forall n \in \mathbb{N}, \quad S_{\theta}(y)(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1} + O(x^n)$$



The Stokes Phenomenon in the Laplace Plane (1)

Analytic continuation of a sum, Stokes directions

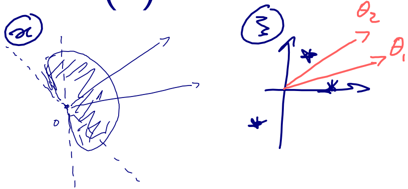
- The sum $\mathcal{S}_\theta(y)$, as a solution of L, can be analytically continued around the origin



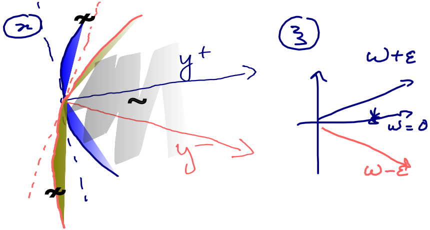
- The analytic continuation remains $\sim y(x)$ in a domain of opening $> \pi$
- …but the asymptotic expansion suddenly changes when crossing a **Stokes direction**

The Stokes Phenomenon in the Laplace Plane (2)

Variing the direction of summation, singular directions



- The sums $\mathcal{S}_\theta(y)$, $\theta \in (\theta_1, \theta_2)$ obtained by variing θ **continuously** — when possible — patch together



- The sums $y^+ = \mathcal{S}_{\omega-\varepsilon}(y)$ and $y^- = \mathcal{S}_{\omega+\varepsilon}(y)$ on both sides of a **Stokes direction** ω are (usually) different

  …but have the same asymptotic expansion!

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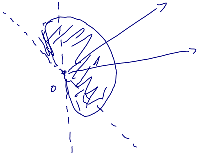

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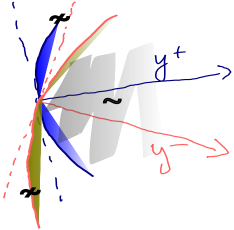

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  …but have the same asymptotic expansion!

- Same asymptotics $\Rightarrow$ $y^+(x) - y^-(x)$ is **exponentially small** "on the whole half-plane"

- Singular directions are those s.t. $e^{-\alpha'/x} \lll e^{-\alpha/x}$ for some exp parts $e^{-\alpha/x}$, $e^{-\alpha'/x}$

  I.e., $\omega = \arg(\alpha' - \alpha)$

Stokes Matrices

Choose a basis $Y = (y_1, \ldots, y_r)$ of formal solutions $\quad y_i(x) = e^{\alpha_i/x} x^{\lambda_i} F_i(x, \log x)$
a singular direction ω of L

Let $Y^\pm$ be the sums of Y to the left/right. (Define $\mathcal{S}_\omega(e^{-\alpha/x} z(x)) = e^{-\alpha/x} \mathcal{S}_\omega(z(x))$.)

Both Y^+ and Y^- are bases of analytic solutions of L on a common domain.

Definition. The **Stokes matrix** of L in the direction ω is the matrix of Y^+ in the basis Y^-:

$$Y^+ = Y^-(I + C)$$

Remark. This definition already gives an algorithm:

- Compute the formal Borel transform $\hat{Y}$ of Y
- Compute its analytic continuation to $[0, e^{i(\omega \pm \varepsilon)} \infty)$ numerically
- Compute the Laplace integrals numerically
- Compare

The Equation in the Borel Plane

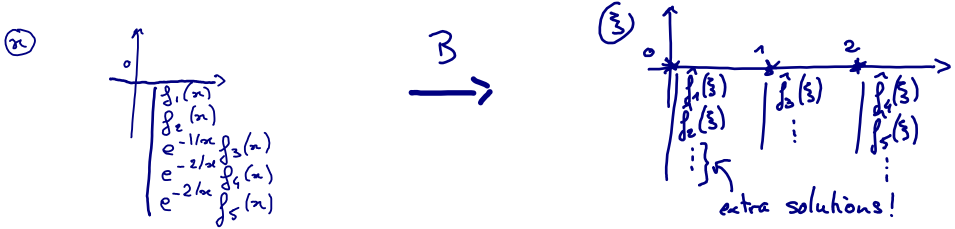
Definition. Formal Borel transform with an exponential part:

$$\mathcal{B}(e^{-\alpha/x} f(x)) = \hat{f}(x - \alpha) \qquad \text{where} \quad \hat{f} = \mathcal{B}(f)$$

Lemma. Given $L \in \mathbb{Q}[x]\langle d/dx \rangle$, one can find a differential operator $\hat{L} \in \mathbb{Q}[\xi]\langle d/d\xi \rangle$ such that the Borel transforms of solutions of L are solutions of $\hat{L}$.

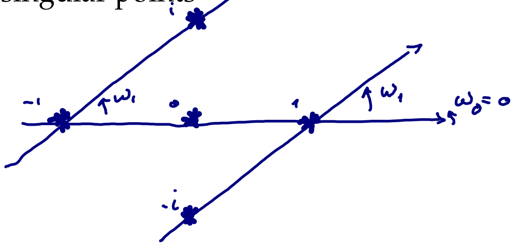
Proposition.

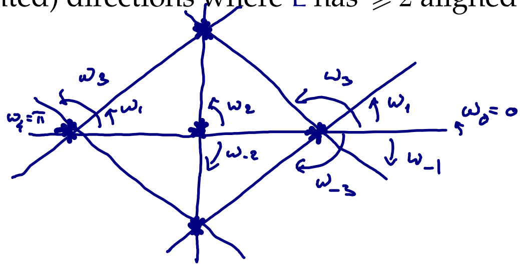
- The finite **singular points** of $\hat{L}$ are the α such that $e^{-\alpha/x}$ is an exp part of L (incl. 0).
- These are **regular** singular points.



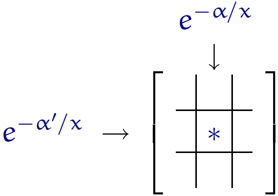
The Stokes Phenomenon in the Borel Plane (1)

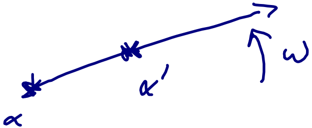
- The **singular directions** of L are the (oriented) directions where $\hat{L}$ has $\geqslant 2$ aligned singular points





- Contributions to the Stokes matrix in the direction ω:

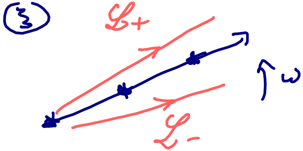




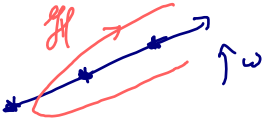
The Stokes Phenomenon in the Borel Plane (2)
Contribution of a singular point

Let y be one of the basis elements.

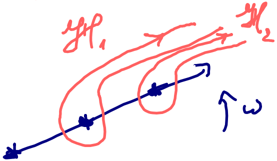
To compute the corresponding column, we need to express y^+ in the basis y^-.



$$y^+ - y^- = \int_{\mathcal{L}^+} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi - \int_{\mathcal{L}^-} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$



$$= \int_{\mathcal{H}} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$



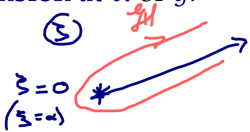
$$= \int_{\mathcal{H}_1} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi + \int_{\mathcal{H}_2} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi + \cdots$$

The Stokes Phenomenon in the Borel Plane (3)

Connection-to-Stokes formulae

We are left with Laplace integrals on Hankel contours enclosing a single α'.

These can be computed by termwise integration of the **local expansion at α** of $\hat{y}$:



$$\hat{y}(\alpha' + \zeta) = \zeta^{\lambda} \sum_{k=0}^{r} \sum_{n=0}^{\infty} c_{n,k}\, \zeta^{n} \log(\zeta)^{k}$$

$$\int_{\mathcal{H}} \hat{y}(\alpha + \zeta)\, e^{-(\alpha+\zeta)/x}\, d\zeta = \sum_{k=0}^{r} \sum_{n=0}^{\infty} c_{n,k}\, e^{-\alpha/x} \underbrace{\int_{\mathcal{H}} \zeta^{\lambda+n} \log(\zeta)^{k} e^{-\zeta/x}\, d\zeta}$$

$$= \frac{d^{k}}{d\lambda^{k}} \frac{2\pi i\, (x\, e^{-\pi i})^{\lambda+n-1}}{\Gamma(-\lambda-n)}$$

$$= x^{\lambda+n-1} \times (\text{explicit polynomial in } \log(x))$$

- Compute enough terms of the expansion of $y^{+} - y^{-}$
- Equate the coefficients of $e^{-\beta/x} x^{\mu} \log(x)^{k}$ to write it in the basis Y^{-}

Summary

Algorithm (sketch). *Input*: L, ω *Output*: the Stokes matrix in the direction ω

Initialize an $r \times r$ matrix $S := I$

For $y = Y_j = e^{-\alpha/x}(\dots)$ in a basis Y of formal solutions of L

For each singular point α' of $\hat{L}$ with $\arg(\alpha' - \alpha) = \omega$

Solve the equation $\hat{L}(\hat{y}) = 0$ numerically to obtain the series expansion **at α'** of the analytic continuation $\hat{y}$

Deduce the coordinates of $y^+ - y$ in the basis Y^- using the **previous slide**

Add the resulting coordinate vector to column j of S

Return S

Need:

- Connection between **regular singular** points
- Computation of $1/\Gamma$ and its derivatives [→ Johansson 2023]
- Some elementary functions; some formal operations on diff. operators and formal solutions

Removing Redundancies (1)

Computing the Stokes matrices in all directions

Fix for each α:

- a basis $Y_{[\alpha]}$ of the space $V_{[\alpha]}$ of formal solutions of L of exponential part $e^{-\alpha/x}$
- a basis $\hat{Y}_{[\alpha]}$ of the space $\hat{V}_{[\alpha]}$ of local solutions of $\hat{L}$ at α

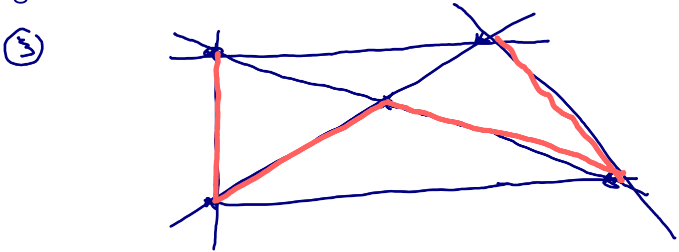
Compute the matrices:

- For each α, of the map $\begin{array}{ccc} V_{[\alpha]} & \to & \hat{V}_{[\alpha]} \\ y & \mapsto & \hat{y} \end{array}$ **Borel transform matrix** B_α
- For each α', of the map $\begin{array}{ccc} \hat{V}_{[\alpha]} & \to & V_{[\alpha]} \\ \hat{y} & \mapsto & \int_{\mathcal{H}} \hat{y}(\zeta)\, e^{-\zeta/x}\, d\zeta \end{array}$ **Connection-to-Stokes matrix** $T_{\alpha,\alpha'}$
- For each pair (α, α'), of 'the' an. cont. map $\hat{V}_{[\alpha]} \to \hat{V}_{[\alpha']}$ **Connection matrix** $L_{\alpha'}$

Fact. The block (α, α') of the Stokes matrix in the direction $\arg(\alpha' - \alpha)$ is $L_{\alpha'}\, T_{\alpha,\alpha'}\, B_\alpha$.

Removing Redundancies (2)

Computing all connection matrices



- Compute the connection matrices along a **spanning tree** of the singular points of $\hat{L}$ as before (numerical integration of th ODE)
- Pick known $T_{\alpha,\alpha'}$, $T_{\alpha',\alpha''}$ s.t. the triangle $(\alpha, \alpha', \alpha'')$ contains no other singular point; compute $T_{\alpha,\alpha''}$ as

$$T_{\alpha,\alpha''} = \tilde{T}_{\alpha',\alpha''} \tilde{T}_{\alpha,\alpha'}$$

  after incorporating correcting factors to get the correct branch
- Repeat

Conclusion

Summary.

Stokes matrices of LODE of pure level 1 are computable in practice

$\longrightarrow$ code available

$\longrightarrow$ roughly as fast as regular singular connection

$\longrightarrow$ rigorous error bounds

Question.

Does this approach generalize to multiple levels?

(The "direct" algorithm does, using, e.g., accelero-summation.)