Truncation Bounds for Differentially Finite Series

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Journées FastRelax

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http://marc.mezzarobba.net/papers/Mezzarobba_AdaptiveBounds.pdf

prepared with GNU $\mathrm{T_EX_{MACS}}$

The Problem

$$\sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\text{known}} + \underbrace{\sum_{n=N}^{\infty} u_n z^n}_{|\cdot| \leqslant ?}$$

- $u(z)$ given by

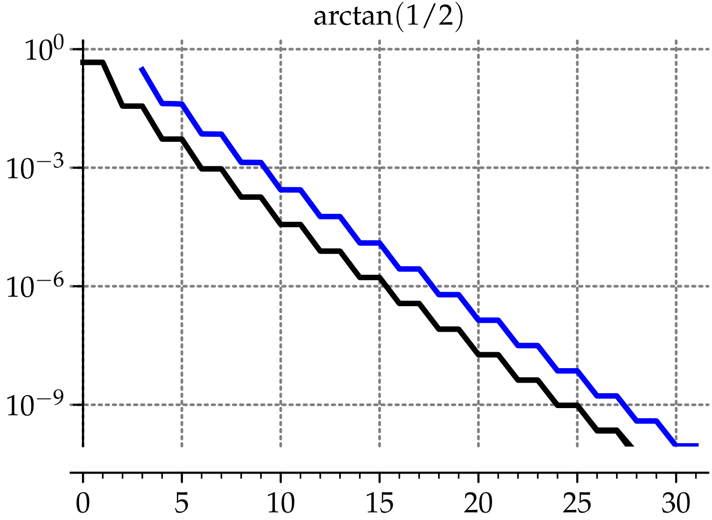
$$\begin{cases} \text{differential operator} & L \in \mathbb{C}[z]\langle {}^{d}/_{dz} \rangle \qquad \text{s.t.} \quad L \cdot u = 0 \\ \text{initial values} & u_0, \ldots, u_{r-1} \end{cases}$$

- More generally: regular singular points

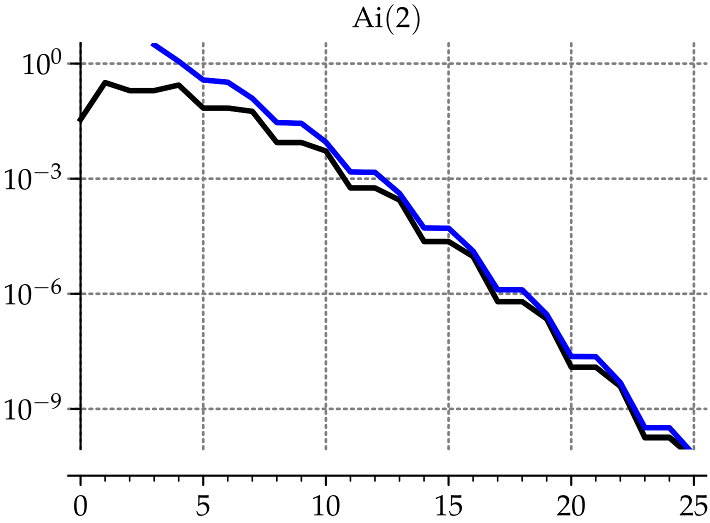
  ↳ logarithms, non-integer exponents

- Goal: accurate bounds **in practice**, at reasonable cost

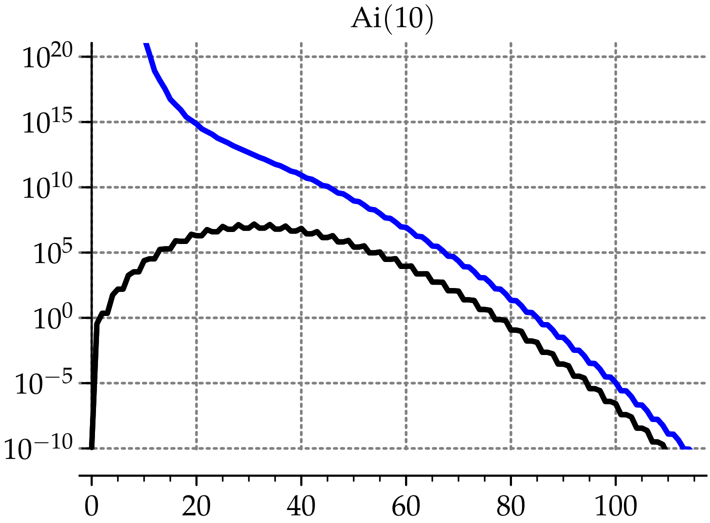
Results



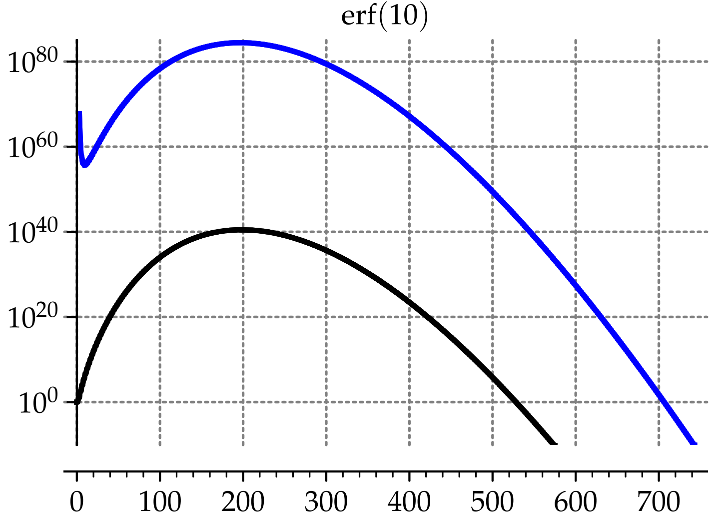
Results



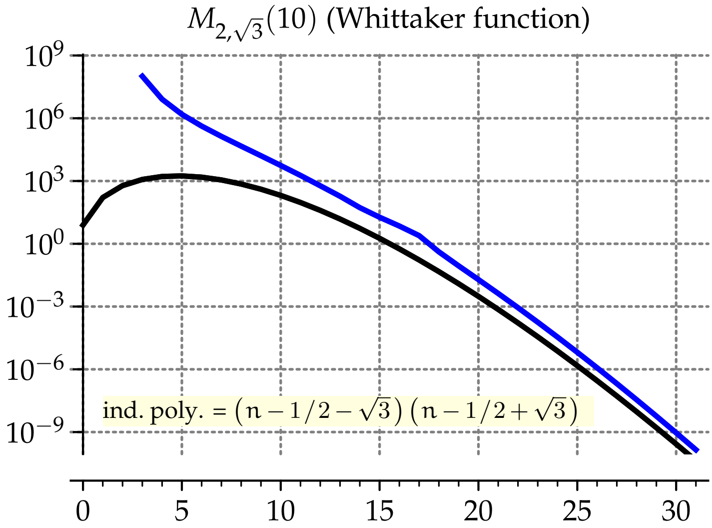
Results



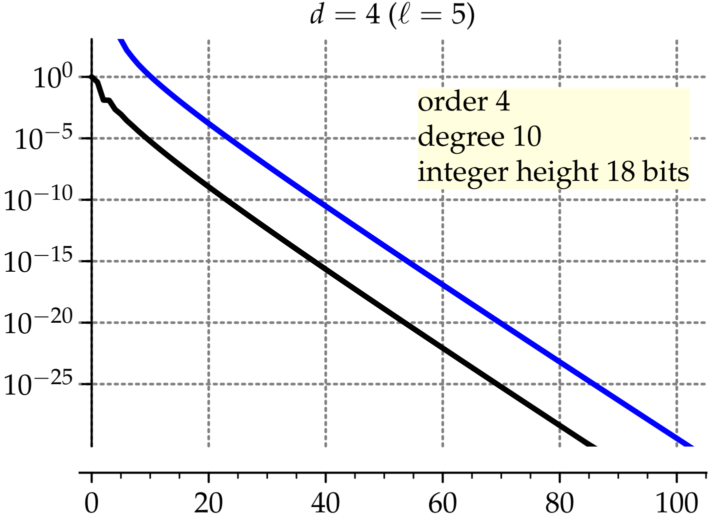
Results



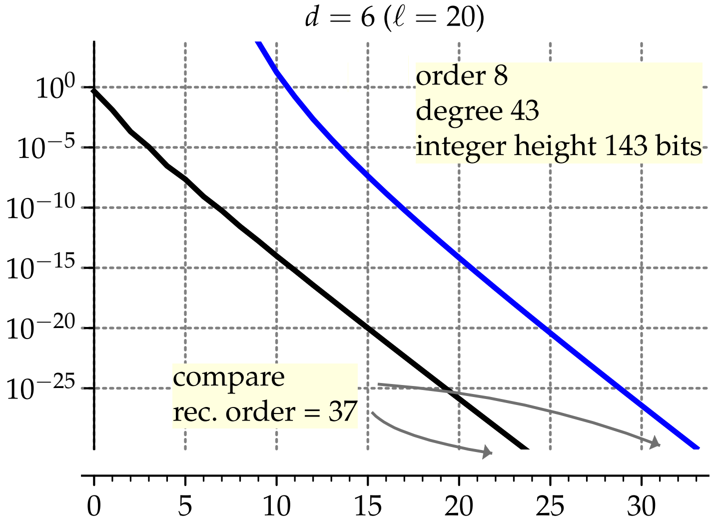
Results


$M_{2,\sqrt{3}}(10)$ (Whittaker function)
10^9
10^6
10^3
10^0
10^{-3}
10^{-6}
10^{-9}
ind. poly. = $\left(n - 1/2 - \sqrt{3}\right)\left(n - 1/2 + \sqrt{3}\right)$
0
5
10
15
20
25
30

Results


$d = 4$ ($\ell = 5$)
order 4
degree 10
integer height 18 bits
10^0
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
0
20
40
60
80
100

Results



$d = 6$ ($\ell = 20$)
order 8
degree 43
integer height 143 bits
compare
rec. order = 37
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
0
5
10
15
20
25
30

Related Work

Cauchy, ~1840 – Majorant series

Moore 1962, ... – Interval enclosure methods

Neher 1999, 2001 – Eff$^{\text{ve}}$ maj. for LODE $\to$ interval Taylor methods

van der Hoeven 1999, 2001, 2003 – Similar, $\to$ high-prec. comput.

Warne *et al.* 2006 – Polynomial differential equations

M. & Salvy 2010 – "Asymptotic" tightness for sequences

...

- None covers regular singular points

  (**vdH2001** sketches an adaptation of **vdH1999**)
- Tightness and efficiency issues (esp. with large equations)

Two Analogies

Tails of the exponential series

$$\left|\sum_{n\geqslant N}\frac{\zeta^n}{n!}\right| \leqslant \frac{|\zeta|^N}{N!}\sum_{n=0}^{\infty}\frac{N!}{(N+n)!}|\zeta|^n \leqslant \underbrace{e^{|\zeta|}}_{\text{“worst case” of function, indep. of } N}\ \underbrace{\frac{|\zeta|^N}{N!}}_{\text{first neglected term}}$$

Residuals of linear systems

$A\,x = b \qquad A \in GL_n(\mathbb{C}), \quad \|A^{-1}\| \leqslant \mathbf{M}$

$A\,\tilde{x} = \tilde{b}$ residual ($\approx$ 1st negl. term)

$\|x - \tilde{x}\| \leqslant \mathbf{M} \cdot \underbrace{\|b - \tilde{b}\|}_{\text{known}}$

$\tilde{x}$: computed approx.

Quantity playing the role of $\mathbf{M}$ *when* A *is a differential operator?*

The Method of Majorants

[Cauchy 1842]

- Instead of directly bounding $|\sum_{n \geqslant N} u_n \zeta^n|$,
  compute a **majorant series**:

$$\sum \hat{u}_n z^n \in \mathbb{R}_{\geqslant 0}[[z]] \qquad \text{s.t.} \qquad \forall n, \quad |u_n| \leqslant \hat{u}_n$$

- To do that, "replace" L with a simple **model equation**:

$$L(z, {}^{d}/_{dz}) \cdot u = 0 \qquad \underset{\text{“bounded by”}}{\ll} \qquad \underset{\text{for us: always 1st order}}{\hat{u}'(z) - \hat{a}(z)\, \hat{u}(z) = 0}$$

- Solve the model equation and study the solutions:

$$\hat{u}(z) = \exp \int^{z} \hat{a}(w)\, dw \qquad\qquad \left| \sum_{n=N}^{+\infty} u_n z^n \right| \leqslant \sum_{n=N}^{+\infty} \hat{u}_n |z|^n \leqslant \cdots$$

Setting

(power series case)

$$\underbrace{[\theta^r p_r(z) + \cdots + \theta\, p_1(z)]}_{P(z,\theta)} \cdot u(z) = 0 \qquad \theta = z\frac{d}{dz}$$

$p_r(0) \neq 0$ (ordinary/reg. sing.)

$$u(z) = \sum_{n=0}^{\infty} u_n z^n$$

$$\tilde{u}(z) = \sum_{n=0}^{N-1} u_n z^n$$

Question: **bound** $\mathbf{u}(z) - \tilde{\mathbf{u}}(z)$

Residual: $P(z,\theta)\cdot(\tilde{u} - u) = P(z,\theta)\cdot\tilde{u} = q(z)$

$$q(z) = \square\, z^N + \cdots + \square\, z^{N+s-1}$$

The “Unbounded” Recurrence

$$\left[\theta^r p_r(z) + \theta^{r-1} p_1(z) \quad + \cdots + p_0(z)\right] \cdot (\tilde{u} - u)(z) \quad = q(z)$$

The "Unbounded" Recurrence

$$\left[\theta^r \qquad + \theta^{r-1}\underbrace{\frac{p_{r-1}(z)}{p_r(z)}}_{a_{r-1}(z)} + \cdots + \underbrace{\frac{p_0(z)}{p_r(z)}}_{a_0(z)}\right] \cdot \underbrace{p_r\,(\tilde{u}-u)(z)}_{y(z)} = q(z)$$

crucial step(!)

The "Unbounded" Recurrence

$$\underbrace{\left[\theta^r \qquad + \theta^{r-1} \underbrace{\frac{p_{r-1}(z)}{p_r(z)}}_{a_{r-1}(z)} + \cdots + \underbrace{\frac{p_0(z)}{p_r(z)}}_{a_0(z)}\right]}_{= \sum_{k=0}^{r} \theta^k \sum_{j=0}^{\infty} a_{k,j} z^j =: \sum_{j=0}^{\infty} Q_j(\theta)\, z^j} \cdot \underbrace{p_r\,(\tilde{u} - u)(z)}_{y(z)} = q(z)$$

crucial step(!)

The "Unbounded" Recurrence

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crucial step(!)

$$\hookrightarrow \quad \left[\sum_{j=0}^{\infty} Q_j(n)\, S_n^{-j}\right] \cdot (y_n)_{n\in\mathbb{Z}} = (q_n)_{n\in\mathbb{Z}}$$

finite sequence

The "Unbounded" Recurrence

$$\underbrace{\left[\theta^r + \theta^{r-1}\underbrace{\frac{p_{r-1}(z)}{p_r(z)}}_{a_{r-1}(z)} + \cdots + \underbrace{\frac{p_0(z)}{p_r(z)}}_{a_0(z)}\right]}_{= \sum_{k=0}^{r} \theta^k \sum_{j=0}^{\infty} a_{k,j} z^j =: \sum_{j=0}^{\infty} Q_j(\theta)\, z^j} \cdot \underbrace{p_r\,(\tilde{u} - u)(z)}_{y(z)} = q(z)$$

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$$\hookrightarrow \quad \left[\sum_{j=0}^{\infty} Q_j(n)\, S_n^{-j}\right] \cdot (y_n)_{n\in\mathbb{Z}} = (q_n)_{n\in\mathbb{Z}}$$

finite sequence

deg $\leqslant r-1$

$$Q_0(n)\, y_n = q_n - \sum_{j=1}^{\infty} Q_j(n)\, y_{n-j}$$

indicial polynomial, deg $= r$

The Majorant Equation (I)

$$Q_0(n)\,y_n = q_n - \sum_{j=1}^{\infty} Q_j(n)\,y_{n-j}$$

$$y_n = \frac{1}{n}\left(\underbrace{\frac{n\,q_n}{Q_0(n)}} - \sum_{j=1}^{\infty} \underbrace{\frac{n\,Q_j(n)}{Q_0(n)}} y_{n-j}\right)$$

bounded for large n

IF for $n \geqslant n_0$

(a) $\left|\frac{n\,q_n}{Q_0(n)}\right| \leqslant \hat{q}_n$

(b) $\left|\frac{n\,Q_j(n)}{Q_0(n)}\right| \leqslant \hat{a}_j \quad (j \geqslant 1)$

THEN for $n \geqslant n_0$

$$|y_n| \leqslant \frac{1}{n}\left(\hat{q}_n + \sum_{j=0}^{\infty} \hat{a}_j\,|y_{n-j}|\right)$$

The Majorant Equation (I)

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bounded for large n

$$\boxed{\hat{y}_n = \frac{1}{n}\left(\hat{q}_n + \sum_{j=0}^{\infty} \hat{a}_j\, \hat{y}_{n-j} \right)}$$

IF for $n \geqslant n_0$

(a) $\left| \dfrac{n\, q_n}{Q_0(n)} \right| \leqslant \hat{q}_n$

(b) $\left| \dfrac{n\, Q_j(n)}{Q_0(n)} \right| \leqslant \hat{a}_j \quad (j \geqslant 1)$

for $n < n_0$

(c) $|y_n| \leqslant \hat{y}_n$

THEN for $n \geqslant n_0$

$$|y_n| \leqslant \frac{1}{n}\left(\hat{q}_n + \sum_{j=0}^{\infty} \hat{a}_j\, |y_{n-j}| \right)$$

for all n,

$|y_n| \leqslant \hat{y}_n$

The Majorant Equation (II)

$$\boxed{\hat{y}_n = \frac{1}{n}\left(\hat{\mathbf{q}}_{\mathbf{j}} + \sum_{j=0}^{\infty} \hat{\mathbf{a}}_{\mathbf{j}}\,\hat{y}_{n-j}\right)} \qquad \longrightarrow \qquad [\theta - \hat{\mathbf{a}}(\boldsymbol{z})] \cdot \hat{y}(z) = \hat{\mathbf{q}}(\boldsymbol{z})$$

$$\hat{y}(z) = \hat{h}(z)\left(\mathbf{c} + \int_0^z \frac{w^{-1}\,\hat{q}(w)}{\hat{h}(w)}\,dw\right), \quad \hat{h}(z) = \exp\int_0^z w^{-1}\,\hat{a}(w)\,dw$$

It remains to choose $\mathbf{n}_0$, $\hat{\mathbf{a}}(\boldsymbol{z})$, $\hat{\mathbf{q}}(\boldsymbol{z})$ and $\mathbf{c}$.

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(b) $\left|\dfrac{n\,Q_j(n)}{Q_0(n)}\right| \leqslant \hat{a}_j \quad (j \geqslant 1)$

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for $n \geqslant \mathbf{n}_0$

(a) $\left|\frac{n\, q_n}{Q_0(n)}\right| \leqslant \hat{\mathbf{q}}_{\mathbf{n}}$ ✓ finitely many q_n

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for $n < n_0$

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for $n \geqslant n_0$

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for $n < n_0$

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Summary

$u_0, \dots, u_{N-1}$ $\qquad$ $P(z,\theta)\cdot u(z) = 0$

$$\sum_{n=N}^{\infty} u_n z^n \underset{\text{term-wise}}{\ll} \underbrace{\hat{g}(z)\,\hat{h}(z)}_{\substack{\approx\text{ “worst”}\\ \text{solution}}} \left(\mathrm{c}\cancel{\mathrm{s}}\mathrm{t} + \underbrace{\int \frac{w^{-1}\,\hat{q}(w)}{\hat{h}(w)}\,dw}_{\approx\text{ residual}}\right)$$

- Easy to compute in low-precision interval arithmetic
- Quite tight even for small N and complicated $P(z,\theta)$

See the paper for

- Regular singular points
- How to compute $\hat{a}(z)$
- A priori bounds
- Derivatives, ...