



# The expected signature of Brownian motion stopped on the boundary of a circle has finite radius of convergence

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## ABSTRACT

The expected signature is an analogue of the Laplace transform for probability measures on rough paths. A key question in the area has been to identify a general condition to ensure that the expected signature uniquely determines the measures. A sufficient condition has recently been given by Chevyrev and Lyons and requires a strong upper bound on the expected signature. While the upper bound was verified for many well-known processes up to a deterministic time, it was not known whether the required bound holds for random time. In fact, even the simplest case of Brownian motion up to the exit time of a planar disc was open. For this particular case we answer this question using a suitable hyperbolic projection of the expected signature. The projection satisfies a three-dimensional system of linear PDEs, which (surprisingly) can be solved explicitly, and which allows us to show that the upper bound on the expected signature is *not* satisfied.

## Contents

1. Introduction	1
2. Differentiability of the development of expected signature	5
3. A polar decomposition for the development	7
4. ODE for $A_\lambda, B_\lambda, C_\lambda$	8
5. Solving the ODE for $A_\lambda, B_\lambda, C_\lambda$	10
6. Conclusion	12
Appendix	13
References	15

## 1. Introduction

Let a probability measure on a subset of the real line have moments of all orders. Under which conditions do these moments pin down the probability measure uniquely? This is the well-studied *moment problem*. When the subset is compact, the answer is always affirmative. In the noncompact case uniqueness is more delicate (see [14]).

In stochastic analysis one is usually concerned with measures on some space of paths, the prime example being *Wiener measure* on the space of continuous functions. It turns out that for many purposes a good replacement for ‘monomials’ in this setting are the iterated integrals of paths. The collection of all of these integrals is called the *iterated-integrals signature*.

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For smooth paths  $X : [0, T] \rightarrow \mathbb{R}^d$  (and with respect to a time horizon  $T > 0$ ), it is defined, using Riemann–Stieltjes integration, as

$$\begin{aligned} S(X)_{0,T} := & 1 + \int_0^T dX_s + \int_0^T \int_0^{r_2} dX_{r_1} \otimes dX_{r_2} \\ & + \int_0^T \int_0^{r_3} \int_0^{r_2} dX_{r_1} \otimes dX_{r_2} \otimes dX_{r_3} + \cdots \in T((\mathbb{R}^d)) := \prod_{n=0}^{\infty} (\mathbb{R}^d)^{\otimes n}. \end{aligned} \quad (1)$$

It is well known (see [1] and references therein) that

- $S(X)_{0,T} \in G$ , where  $G \subset T((\mathbb{R}^d))$  is the group of *grouplike elements*;
- $S(X)_{0,T}$  completely characterizes the path  $X$  up to reparametrization and up to tree-likeness.

Let  $X : \Omega \times [0, T] \rightarrow \mathbb{R}^d$  now be a stochastic process. For fixed  $\omega \in \Omega$ ,  $t \mapsto X_t(\omega)$  is usually *not* smooth, so that we have to assume that the stochastic process possesses a ‘reasonable’ integration theory.

In particular assume that integrals of the form  $\int g(X_s) dX_s$ , exist for a large class of functions  $g \in C(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^n))$  and that the fundamental theorem of calculus holds:

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_{x_i} f(X_s) dX_s^i.$$

Examples include Brownian motion with Stratonovich integration and fractional Brownian motion with Hurst parameter strictly larger than 1/2 with Young integration. For further examples, see [4].

The iterated-integrals signature  $S(X)_{0,T}$  defined by the expression (1) — now, using the given integration theory — is then a random variable. Let us assume that we can take its expectation level by level:

$$\mathbb{E} \left[ \left\| \int_0^T \int_0^{r_n} \cdots \int_0^{r_2} dX_{r_1} \otimes \cdots \otimes dX_{r_n} \right\| \right] < +\infty.$$

The choice of norm in fact does not matter here but we will equip  $\mathbb{R}^d$  with the Euclidean norm and the projective norm  $\|\cdot\|$  on  $(\mathbb{R}^d)^{\otimes n}$  throughout this paper. We can then define expected signature level by level:

$$\begin{aligned} \text{ExpSig}(X)_T &:= \mathbb{E}_{\mathfrak{p}}[S(X)_{0,T}] \\ &:= \sum_{n=0}^{\infty} \mathbb{E} \left[ \int_0^T \int_0^{r_n} \cdots \int_0^{r_2} dX_{r_1} \otimes \cdots \otimes dX_{r_n} \right] \in T((\mathbb{R}^d)), \end{aligned} \quad (2)$$

where  $\mathbb{E}_{\mathfrak{p}}$  denotes the expectation level by level ( $\mathfrak{p}$  for ‘projective’) of a  $G$ -valued random variable. The question arises:

$\text{ExpSig}(X)$  completely characterize the law of  $X$

The correspondence between  $X$  and  $S(X)_{0,T}$  was studied in [1] and references therein. Here we will focus on

$\text{ExpSig}(X)_T$  completely characterize the law of  $S(X)_{0,T}$  on  $G$

A sufficient condition for this to be the case is given in [2]: if  $\text{ExpSig}(X)_T$  has infinite radius of convergence, that is,

$$\sum_{n \geq 0} \|\text{proj}_n \text{ExpSig}(X)_T\| \lambda^n < +\infty, \quad (3)$$

for all  $\lambda > 0$  then the law of  $S(X)_{0,T}$  on  $G$  is the unique law with this (projective) expected value. Here  $\text{proj}_n : T((\mathbb{R}^d)) \rightarrow (\mathbb{R}^d)^{\otimes n}$  denotes projection onto tensors of length  $n$ .

Let us give two examples. Let  $\mu$  be a probability measure on  $\mathbb{R}$  having all moments and define

$$a_n := \int x^n \mu(dx).$$

Consider the stochastic process  $X_t := tZ$ , where  $Z$  is distributed according to  $\mu$ . Since  $X$  is smooth, its signature is well defined and actually has the simple form

$$S(X)_{0,T} = 1 + TZ + \frac{T^2}{2!}Z^2 + \frac{T^3}{3!}Z^3 + \dots$$

Then

$$\text{ExpSig}(X)_T = 1 + Ta_1 + \frac{T^2}{2!}a_2 + \frac{T^3}{3!}a_3 + \dots,$$

and a sufficient condition for  $\sum_n a_n T^n \lambda^n / n!$  to have infinite radius of convergence is  $|a_n| \leq C^n$ , for some  $C > 0$ .<sup>†</sup> Then [2, Proposition 6.1] applies, and the law of  $S(X)_{0,T}$  on  $G$  is uniquely determined by these moments.

Consider now the expected signature of a standard Brownian motion  $B$  calculated up to some fixed time  $T > 0$ . It is known (see for example [9, Proposition 4.10]) that

$$\text{ExpSig}(B)_T = \exp\left(\frac{T}{2} \sum_{i=1}^d e_i \otimes e_i\right).$$

It follows that

$$\|\text{proj}_{2n} \text{ExpSig}(B)\| = \left\| \frac{T^n}{2^n n!} \left( \sum_{i=1}^d e_i \otimes e_i \right)^n \right\| \leq \frac{d^n T^n}{2^n n!},$$

and hence

$$\sum \|\text{proj}_n \text{ExpSig}(B)_T\| \lambda^n < +\infty, \quad (4)$$

for any  $\lambda > 0$ . Again, by [2, Proposition 6.1], the law of  $S(B)_{0,T}$  is uniquely determined by  $\text{ExpSig}(B)_T$ . The condition (4) in fact holds more generally when  $B$  is a Gaussian process or a Markov process; see [2, 3, 12] for further details.

In this paper we study properties of the expected signature, not up to deterministic time  $T$ , but up to a *stopping time*  $\tau$ . Concretely, we consider the Brownian motion  $B^z$  in  $\mathbb{R}^2$  started at some point  $z$  in the unit circle  $\mathbb{D} := \{z \in \mathbb{R}^2 : |z|_2 \leq 1\}$ , and stopped at hitting the boundary, that is,

$$\tau := \inf \{t \geq 0 : B_t^z \in \partial\mathbb{D}\}, \quad (5)$$

with  $|\cdot|_2$  denoting the Euclidean norm on  $\mathbb{R}^2$ . In the notation introduced above, we are interested in

$$\Phi(z) := \text{ExpSig}(X^z)_\infty,$$

where  $X_t^z := B_{t \wedge \tau}^z$ . In [8] it was shown that for every  $n \in \mathbb{N}$  and  $n \geq 2$ , the  $n$ th term of  $\Phi$  satisfies the following PDE:

$$\Delta(\text{proj}_n(\Phi(z))) = -2 \sum_{i=1}^d e_i \otimes \frac{\partial \text{proj}_{n-1}(\Phi(z))}{\partial z_i} - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \text{proj}_{n-2}(\Phi(z)), \quad (6)$$

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<sup>†</sup>The condition  $|a_n| \leq C^n$  is of course more than enough in the classical moment problem to have uniqueness for the law  $\mu$  on  $\mathbb{R}$  [13, Example X.6.4].

with the boundary condition that for each  $|z|_2 = 1$ ,

$$\text{proj}_n(\Phi(z)) = \begin{cases} 0, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

Additionally, one has  $\text{proj}_0(\Phi(z)) = 1$  and  $\text{proj}_1(\Phi(z)) = 0$  for all  $z \in \mathbb{D}$ . Using this, they were able to obtain the bound  $\|\text{proj}_n(\Phi(z))\| \leq C^n$  for some  $C > 0$  [8, Theorem 3.6]. This bound is *not* enough to decide whether the radius of convergence for  $\text{ExpSig}(X^z)_\infty$  is infinite or not, but it is enough to deduce that  $\text{ExpSig}(X^z)_\infty$  has radius of convergence strictly larger than 0. In this work we show that the radius of convergence is indeed finite.

Recall from [2, Proposition 6.1] that if  $A, B$  are  $G$ -valued random variables such that  $\mathbb{E}_p[A] = \mathbb{E}_p[B]$  and  $\mathbb{E}_p[A]$  has an infinite radius of convergence, then  $A \stackrel{D}{=} B$ . Our main theorem, proven in Section 6, is the following.

**THEOREM.** *The expected signature  $\Phi(0) = \text{ExpSig}(X^0)_\infty = \mathbb{E}_p[S(X^0)_{0,\infty}]$  of a two-dimensional Brownian motion stopped upon exiting the unit disk has a finite radius of convergence.*

Inspired by [6, 10], we consider the map  $M \in L(\mathbb{R}^2, L(\mathbb{R}^3, \mathbb{R}^3))$  defined by

$$M : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 0 \end{pmatrix}. \quad (7)$$

By [6, Lemma 3.1], one has

$$\|M\|_{L(\mathbb{R}^2, L(\mathbb{R}^3, \mathbb{R}^3))} = \sup_{|u|_2=1, |v|_2=1} |M(u)v|_2 = 1. \quad (8)$$

We may extend  $M$  as a continuous linear map on the  $k$ -times tensor product  $(\mathbb{R}^2)^{\otimes k}$ , by the following relation:

If  $k = 0$ , we take by convention that  $(\mathbb{R}^d)^{\otimes 0} := \mathbb{R}$  and define

$$M(v) = 1 \quad \forall v \in \mathbb{R}. \quad (9)$$

For  $k \geq 1$ , if  $v_1, \dots, v_k \in \mathbb{R}^2$ , then

$$M(v_1 \otimes \dots \otimes v_k) = M(v_1) \cdots M(v_k). \quad (10)$$

Using (8) and the definition of projective norm, we see that

$$\|M(\text{proj}_n(\Phi(z)))\|_{L(\mathbb{R}^3, \mathbb{R}^3)} \leq \|\text{proj}_n(\Phi(z))\|.$$

Therefore, to show the finite radius of convergence theorem, it is sufficient to show that there exists  $\lambda^* \in \mathbb{R}$  such that the matrix

$$\sum_{n=0}^{\infty} \lambda^n M(\text{proj}_n(\Phi(z))) \quad (11)$$

at  $z = 0$  diverges as  $\lambda$  tends to a finite number  $\lambda^*$ .

We proceed as follows. In Section 2, for  $\lambda > 0$  and sufficiently small, we show that the matrix given in (11) acting on  $(0, 0, 1) \in \mathbb{R}^3$  is smooth in  $z$  and solves a certain PDE. Using rotational invariance of Brownian motion, in Section 3 we rewrite said PDE solution in polar coordinates. In Sections 4 and 5, we obtain an explicit solution for the PDE (still, for  $\lambda$  small enough) in terms of Bessel functions. Finally, in Section 6 we show that the solution blows up as  $\lambda \rightarrow \lambda^*$

for some  $\tilde{\lambda} < +\infty$ , proving our main theorem. The Appendix contains some auxiliary results on PDEs.

## 2. Differentiability of the development of expected signature

We first need two technical lemmas which assert that the development of the expected signature is twice differentiable, and satisfies the PDE we expect it to. In Lemma 1 we will adopt the multi-index notation

$$|(\alpha_1, \alpha_2)| = |\alpha_1| + |\alpha_2|, \quad D^{(\alpha_1, \alpha_2)}u(z) = \frac{\partial^{\alpha_1 + \alpha_2} u}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}}(z), \quad (\alpha_1, \alpha_2) \in \mathbb{N}^2.$$

The convergence of the following series:

$$(\lambda M)\Phi(z) = \sum_{n=0}^{\infty} \lambda^n M \text{proj}_n(\Phi(z))$$

for sufficiently small  $\lambda$  has been established in [8, Theorem 3.6].

**LEMMA 1.** *The function  $z \mapsto \text{proj}_n(\Phi(z))$  is twice continuously differentiable. There exists a constant  $C > 0$  such that for all  $n \in \mathbb{N}$ , all  $z \in \mathbb{D}$  and all  $\alpha \in \mathbb{N}^2$  satisfying  $|\alpha| \leq 2$ , one has the bound*

$$\|D^\alpha \text{proj}_n(\Phi(z))\| \leq C^n.$$

Moreover, there exists  $\lambda^* > 0$  such that for all  $\lambda < \lambda^*$

$$z \mapsto \sum_{n=0}^{\infty} \lambda^n M \text{proj}_n(\Phi(z))$$

is twice differentiable in  $z$  and if  $|\alpha| \leq 2$ , then

$$D^\alpha \sum_{n=0}^{\infty} \lambda^n M \text{proj}_n(\Phi(z)) = \sum_{n=0}^{\infty} \lambda^n D^\alpha M \text{proj}_n(\Phi(z)).$$

*Proof.* Let  $m \in \mathbb{N}$ . By Theorem A.5, the function  $z \mapsto \text{proj}_n(\Phi(z))$  is twice continuously differentiable (it is in fact infinitely differentiable on  $\mathbb{D}$ ). By Lemma A.4, there exists  $C > 0$  such that for all  $n \in \mathbb{N}$ :

$$\|\text{proj}_n(\Phi)\|_{W^{m,2}(\mathbb{D})} \leq C^n, \tag{12}$$

where the norm  $\|\cdot\|_{W^{m,2}(\mathbb{D})}$  is the Sobolev norm on the unit disc  $\mathbb{D}$  with respect to the variable  $z$ ,

$$\|u\|_{W^{m,2}(\mathbb{D})} = \max_{|\alpha|=m} \|D^\alpha u\|_{L^2(\mathbb{D})}.$$

Note that as  $\Phi$  is twice continuously differentiable in a strong sense by [8, Theorem 3.2], the derivative  $D^\alpha$  below may be taken in the strong sense.

By [8, Theorem 2.2], which bounds the values of a function  $u$  in terms of the Sobolev norm of  $u$ , there is some constant  $\tilde{C}(2)$  such that for all  $z \in \mathbb{D}$  and  $|\alpha| \leq 2$ ,

$$\begin{aligned} |D^\alpha \text{proj}_n(\Phi(z))| &\leq \tilde{C}(2) \|D^\alpha \text{proj}_n(\Phi)\|_{W^{2,2}(\mathbb{D})} \\ &\leq \tilde{C}(2) \|\text{proj}_n(\Phi)\|_{W^{4,2}(\mathbb{D})} \\ &\leq \tilde{C}(2) C(4)^n. \end{aligned}$$

Since  $M \operatorname{proj}_n(\Phi(z))$  is a linear image of  $\operatorname{proj}_n(\Phi(z))$ , the function  $M \operatorname{proj}_n(\Phi(z))$  is twice continuously differentiable in  $z$ , and moreover, there exists  $c > 0$  such that for all  $z \in \mathbb{D}$ ,

$$|D^\alpha M \operatorname{proj}_n(\Phi(z))| \leq c^n.$$

This bound also allows us to deduce that for  $|\alpha| = 2$ , the series

$$\sum_{n=0}^{\infty} \lambda^n D^\alpha M \operatorname{proj}_n(\Phi(z))$$

converges uniformly and hence the series

$$\sum_{n=0}^{\infty} \lambda^n M \operatorname{proj}_n(\Phi(z))$$

is twice continuously differentiable and the derivatives can be taken inside the infinite summation.  $\square$

LEMMA 2. *There exists  $\lambda^* > 0$  such that if  $\lambda < \lambda^*$ , the function  $F_\lambda$  defined by*

$$F_\lambda(z) = (\lambda M)\Phi(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sum_{n=0}^{\infty} \lambda^n M \operatorname{proj}_n(\Phi(z)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (13)$$

is twice continuously differentiable on  $\mathbb{D}$ , and satisfies

$$\Delta F_\lambda(z) = -2\lambda \sum_{i=1}^2 M e_i \frac{\partial F_\lambda}{\partial z_i}(z) - \lambda^2 \left( \sum_{i=1}^2 (M e_i)^2 \right) F_\lambda(z) \quad (14)$$

with  $F_\lambda(z) = (0, 0, 1)$  for  $z \in \partial\mathbb{D}$ . Here  $(e_1, e_2)$  denotes the canonical basis of  $\mathbb{R}^2$ .

*Proof.* If we apply the linear map  $M$  to the PDE (6), then we have a matrix-valued PDE

$$\Delta(M \operatorname{proj}_n(\Phi(z))) = -2 \sum_{i=1}^2 M e_i \frac{\partial M \operatorname{proj}_{n-1}(\Phi(z))}{\partial z_i} - \left( \sum_{i=1}^2 (M e_i)^2 \right) M \operatorname{proj}_{n-2}(\Phi(z)), \quad (15)$$

together with the boundary condition

$$M \operatorname{proj}_0(\Phi(z)) = I_{3 \times 3}$$

$$M \operatorname{proj}_1(\Phi(z)) = 0_{3 \times 3}.$$

We may multiply both sides with  $\lambda^n$ , sum to infinity and apply to the vector  $(0,0,1)$  to get

$$\begin{aligned} & \sum_{n=0}^{\infty} \lambda^n \Delta(M \operatorname{proj}_n(\Phi(z))) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= -2\lambda \sum_{i=1}^2 M e_i \sum_{n=0}^{\infty} \lambda^n \frac{\partial}{\partial z_i} \operatorname{proj}_n(M\Phi(z)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \lambda^2 \left( \sum_{i=1}^2 (M e_i)^2 \right) F_\lambda(z). \end{aligned}$$

By Lemma 1, each infinite sum converges and we may take the derivatives outside the infinite sum.  $\square$

## 3. A polar decomposition for the development

Let  $x = (x_1, x_2)^t \in \mathbb{R}^2$ . Recall that

$$M(x) = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & 0 \end{pmatrix}.$$

We may consider  $M(x)$  as a linear endomorphism of  $\mathbb{R}^2 \oplus \mathbb{R}$  mapping  $(v, \alpha)$  to  $(\alpha x, \langle x, v \rangle)$ .

LEMMA 3. For any linear map  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$M(R(x)) = (R \oplus 1)M(x)(R^* \oplus 1),$$

where  $R^*$  is the transpose of  $R$ .

*Proof.* Note that

$$\begin{aligned} (R \oplus 1)M(x)(R^* \oplus 1)(v, \alpha) &= (R \oplus 1)M(x)(R^*v, \alpha) = (R \oplus 1)(\alpha x, \langle x, R^*v \rangle) \\ &= (\alpha R(x), \langle x, R^*v \rangle) = (\alpha R(x), \langle Rx, v \rangle) = M(R(x)). \end{aligned} \quad \square$$

In what follows, we will use the notation

$$\Delta_n(0, t) = \{(t_1, \dots, t_n) : 0 < t_1 < \dots < t_n < t\}.$$

COROLLARY 4. Let  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the rotation map

$$z \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} z.$$

Then there exists  $\lambda^* > 0$  such that if  $0 \leq \lambda < \lambda^*$ , for all  $z \in \mathbb{D}$ ,

$$(\lambda M)\Phi(R(z)) = (R \oplus 1)(\lambda M)\Phi(z)(R^* \oplus 1).$$

*Proof.* Brownian motion  $B^{R(z)}$  starting at  $R(z)$  has the same distribution as the rotated Brownian motion  $R(B^z)$ , where  $B^z$  starts from  $z$ . Let  $\text{od}$  denote the Stratonovich differential. Then

$$\begin{aligned} (\lambda M)\Phi(R(z)) &= \sum_{n=0}^{\infty} \lambda^n \mathbb{E}^{R(z)} \left[ \int_{\Delta_n(0, \tau_{\mathbb{D}})} M(\text{od}B_{t_1}^z) \cdots M(\text{od}B_{t_n}^z) \right] \\ &= \sum_{n=0}^{\infty} \lambda^n \mathbb{E}^z \left[ \int_{\Delta_n(0, \tau_{\mathbb{D}})} M(R(\text{od}B_{t_1}^z)) \cdots M(R(\text{od}B_{t_n}^z)) \right]. \end{aligned}$$

By Lemma 3

$$\begin{aligned} &\int_{\Delta_n(0, \tau_{\mathbb{D}})} M(R(\text{od}B_{t_1})) \cdots M(R(\text{od}B_{t_n})) \\ &= \int_{\Delta_n(0, \tau_{\mathbb{D}})} (R \oplus 1)M(\text{od}B_{t_1})(R^* \oplus 1) \cdots (R \oplus 1)M(\text{od}B_{t_n})(R^* \oplus 1). \end{aligned}$$

As  $R$  is orthogonal, we have

$$\begin{aligned} &\int_{\Delta_n(0, \tau_{\mathbb{D}})} M(R(\text{od}B_{t_1})) \cdots M(R(\text{od}B_{t_n})) \\ &= (R \oplus 1) \int_{\Delta_n(0, \tau_{\mathbb{D}})} M(\text{od}B_{t_1}) \cdots M(\text{od}B_{t_n})(R^* \oplus 1). \end{aligned}$$

Therefore,

$$\begin{aligned}
& (\lambda M)\Phi(R(z)) \\
&= (R \oplus 1) \sum_{n=0}^{\infty} \lambda^n \mathbb{E}^z \int_{\Delta_n(0, \tau_{\mathbb{D}})} M(\text{od}B_{t_1}^z) \cdots M(\text{od}B_{t_n}^z)(R^* \oplus 1) \\
&= (R \oplus 1)(\lambda M)\Phi(z)(R^* \oplus 1). \quad \square
\end{aligned}$$

COROLLARY 5. Define the functions  $A_\lambda, B_\lambda, C_\lambda : [0, 1] \rightarrow \mathbb{R}$  by

$$\begin{pmatrix} A_\lambda(r) \\ B_\lambda(r) \\ C_\lambda(r) \end{pmatrix} = F_\lambda(r, 0), \quad (16)$$

where  $F_\lambda$  is the function defined by (13). In polar coordinates, the expression of  $F_\lambda$  reads

$$F(r \cos \theta, r \sin \theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\lambda(r) \\ B_\lambda(r) \\ C_\lambda(r) \end{pmatrix}.$$

Additionally, there exists  $\lambda^* > 0$  such that if  $\lambda < \lambda^*$ , then  $A_\lambda, B_\lambda, C_\lambda$  are twice continuously differentiable functions in the variable  $r$  for all  $r \in [0, 1]$ .

*Proof.* The functions  $A_\lambda, B_\lambda, C_\lambda$  are twice continuously differentiable because  $F_\lambda(r, 0)$  is twice continuously differentiable by Lemma 2. Let  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the rotation of angle  $\theta$ . For  $z = (r \cos \theta, r \sin \theta) = R(r, 0)$ , the definition of  $F_\lambda$  gives

$$F_\lambda(z) = (\lambda M)\Phi(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (\lambda M)\Phi(R(r, 0)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By Corollary 4, one has

$$F_\lambda(z) = (R \oplus 1)(\lambda M)\Phi(R(r, 0))(R^* \oplus 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (R \oplus 1)F_\lambda(r, 0),$$

where

$$R \oplus 1 = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \square$$

#### 4. ODE for $A_\lambda, B_\lambda, C_\lambda$

LEMMA 6. There exists  $\lambda^* > 0$  such that for  $\lambda < \lambda^*$ , the functions  $A_\lambda, B_\lambda, C_\lambda$  defined in Lemma 5 satisfy

$$\begin{aligned}
r^2 A_\lambda''(r) + r A_\lambda'(r) - A_\lambda(r) + \lambda^2 r^2 A_\lambda(r) + 2\lambda r^2 C_\lambda'(r) &= 0 \\
r^2 B_\lambda''(r) + B_\lambda'(r)r - B_\lambda(r) + r^2 \lambda^2 B_\lambda(r) &= 0 \quad (17)
\end{aligned}$$

and

$$C_\lambda'(r) + r C_\lambda''(r) + 2\lambda^2 r C_\lambda(r) + 2\lambda r A_\lambda'(r) + 2\lambda A_\lambda(r) = 0$$

$$A_\lambda(0) = 0, B_\lambda(0) = 0, A_\lambda(1) = 0$$

$$C_\lambda'(0) = 0, B_\lambda(1) = 0, C_\lambda(1) = 1.$$



*Proof.* By Corollary 5,

$$F_\lambda \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta A_\lambda(r) - \sin \theta B_\lambda(r) \\ \sin \theta A_\lambda(r) + \cos \theta B_\lambda(r) \\ C_\lambda(r) \end{pmatrix}. \quad (18)$$

As  $A_\lambda, B_\lambda, C_\lambda$  are twice continuously differentiable for  $\lambda < \lambda^*$ , we may substitute (18) for  $F_\lambda$  into the equation in Lemma 2, which gives for  $r > 0$

$$\begin{aligned} \Delta \begin{pmatrix} F_\lambda^{(1)} \\ F_\lambda^{(2)} \\ F_\lambda^{(3)} \end{pmatrix} &= \begin{pmatrix} \partial_{rr} F_\lambda^{(1)} + \frac{1}{r} \partial_r F_\lambda^{(1)} + \frac{1}{r^2} \partial_{\theta\theta} F_\lambda^{(1)} \\ \partial_{rr} F_\lambda^{(2)} + \frac{1}{r} \partial_r F_\lambda^{(2)} + \frac{1}{r^2} \partial_{\theta\theta} F_\lambda^{(2)} \\ \partial_{rr} F_\lambda^{(3)} + \frac{1}{r} \partial_r F_\lambda^{(3)} + \frac{1}{r^2} \partial_{\theta\theta} F_\lambda^{(3)} \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta A_\lambda''(r) - \sin \theta B_\lambda''(r) + \frac{1}{r} (\cos \theta A_\lambda'(r) - \sin \theta B_\lambda'(r)) \\ \quad + \frac{1}{r^2} (-\cos \theta A_\lambda(r) + \sin \theta B_\lambda(r)) \\ \sin \theta A_\lambda''(r) + \cos \theta B_\lambda''(r) + \frac{1}{r} (\sin \theta A_\lambda'(r) + \cos \theta B_\lambda'(r)) \\ \quad + \frac{1}{r^2} (-\sin \theta A_\lambda(r) - \cos \theta B_\lambda(r)) \\ C_\lambda''(r) + \frac{1}{r} C_\lambda'(r) \end{pmatrix}. \end{aligned} \quad (19)$$

Using the identities

$$\begin{aligned} \frac{\partial}{\partial z_1} &= \frac{\partial r}{\partial z_1} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z_1} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial z_2} &= \frac{\partial r}{\partial z_2} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z_2} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}, \end{aligned}$$

the right-hand side of (14) is

$$\begin{aligned} &\begin{pmatrix} -\lambda^2 \cos \theta A_\lambda(r) + \lambda^2 \sin \theta B_\lambda(r) - 2\lambda \cos \theta C_\lambda'(r) \\ -\lambda^2 \sin \theta A_\lambda(r) - \lambda^2 \cos \theta B_\lambda(r) - 2\lambda \sin \theta C_\lambda'(r) \\ -2\lambda^2 C_\lambda(r) - 2\lambda [\cos^2 \theta A_\lambda'(r) - \cos \theta \sin \theta B_\lambda'(r)] \\ \dots - 2\lambda \left[ \frac{\sin \theta}{r} (\sin \theta A_\lambda(r) + \cos \theta B_\lambda(r)) \right] \\ \dots - 2\lambda [\sin^2 \theta A_\lambda'(r) + \sin \theta \cos \theta B_\lambda'(r)] \\ \dots - 2\lambda \left[ \frac{\cos \theta}{r} \cos \theta A_\lambda(r) - \frac{\cos \theta}{r} \sin \theta B_\lambda(r) \right] \end{pmatrix} \\ &= \begin{pmatrix} -\lambda^2 \cos \theta A_\lambda(r) + \lambda^2 \sin \theta B_\lambda(r) - 2\lambda \cos \theta C_\lambda'(r) \\ -\lambda^2 \sin \theta A_\lambda(r) - \lambda^2 \cos \theta B_\lambda(r) - 2\lambda \sin \theta C_\lambda'(r) \\ -2\lambda^2 C_\lambda(r) - 2\lambda A_\lambda'(r) - \frac{2\lambda}{r} A_\lambda(r) \end{pmatrix}. \end{aligned} \quad (20)$$

Equating (19) and (20) gives the first equation as

$$\begin{aligned} &\cos \theta A_\lambda''(r) - \sin \theta B_\lambda''(r) + \frac{1}{r} (\cos \theta A_\lambda'(r) - \sin \theta B_\lambda'(r)) \\ &\quad + \frac{1}{r^2} (-\cos \theta A_\lambda(r) + \sin \theta B_\lambda(r)) \\ &= -\lambda^2 \cos \theta A_\lambda(r) + \lambda^2 \sin \theta B_\lambda(r) - 2\lambda \cos \theta C_\lambda'(r). \end{aligned}$$

As this holds for all  $\theta$ , we may equate the coefficients of  $\sin \theta$  and  $\cos \theta$  to obtain

$$\begin{aligned} &A_\lambda''(r) + \frac{A_\lambda'(r)}{r} - \frac{1}{r^2} A_\lambda(r) = -\lambda^2 A_\lambda(r) - 2\lambda C_\lambda'(r) \\ &-B_\lambda''(r) - \frac{1}{r} B_\lambda'(r) + \frac{1}{r^2} B_\lambda(r) = \lambda^2 B_\lambda(r) \end{aligned} \quad (21)$$

and from the second equation,

$$\begin{aligned} & \sin \theta A_\lambda''(r) + \cos \theta B_\lambda''(r) + \frac{1}{r}(\sin \theta A_\lambda'(r) + \cos \theta B_\lambda'(r)) \\ & + \frac{1}{r^2}(-\sin \theta A_\lambda(r) - \cos \theta B_\lambda(r)) \\ & = -\lambda^2 \sin \theta A_\lambda(r) - \lambda^2 \cos \theta B_\lambda(r) - 2\lambda \sin \theta C_\lambda'(r) \end{aligned}$$

and therefore,

$$\begin{aligned} A_\lambda''(r) + \frac{1}{r}A_\lambda'(r) - \frac{1}{r^2}A_\lambda(r) &= -\lambda^2 A_\lambda(r) - 2\lambda C_\lambda'(r) \\ B_\lambda''(r) + \frac{B_\lambda'(r)}{r} - \frac{B_\lambda(r)}{r^2} &= -\lambda^2 B_\lambda(r). \end{aligned} \quad (22)$$

Combining (21) and (22) and multiplying the equations throughout by  $r^2$ , we have

$$\begin{aligned} r^2 A_\lambda''(r) + r A_\lambda'(r) - A_\lambda(r) + \lambda^2 r^2 A_\lambda(r) + 2\lambda r^2 C_\lambda'(r) &= 0 \\ r^2 B_\lambda''(r) + B_\lambda'(r)r - B_\lambda(r) + r^2 \lambda^2 B_\lambda(r) &= 0 \\ C_\lambda'(r) + r C_\lambda''(r) + 2\lambda^2 r C_\lambda(r) + 2\lambda r A_\lambda'(r) + 2\lambda A_\lambda(r) &= 0 \end{aligned} \quad (23)$$

for all  $r > 0$ . By continuity of the second derivatives of  $A_\lambda, B_\lambda, C_\lambda$  (see Lemma 5), the equations hold for all  $r \geq 0$ . Again, using the continuity of the second derivatives of  $A_\lambda, B_\lambda, C_\lambda$ , we may substitute  $r = 0$  into (23) to get

$$A_\lambda(0) = B_\lambda(0) = C_\lambda'(0) = 0.$$

Using the boundary conditions for  $z \in \partial\mathbb{D}$  in Lemma 2, we have

$$\begin{pmatrix} A_\lambda(1) \\ B_\lambda(1) \\ C_\lambda(1) \end{pmatrix} = F_\lambda(1, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \square$$

## 5. Solving the ODE for $A_\lambda, B_\lambda, C_\lambda$

LEMMA 7. *Let*

$$\zeta = \sqrt{\frac{-1 + i\sqrt{7}}{2}}, \quad \alpha = \frac{1}{2}\zeta^3 + \zeta, \quad d(\lambda) = \operatorname{Im}(\bar{\alpha}J_0(\lambda\zeta)J_1(\lambda\bar{\zeta})), \quad (24)$$

where  $J_0, J_1$  are the Bessel functions of the first kind. Fix  $\lambda > 0$  such that  $d(\lambda)J_1(\lambda) \neq 0$ . Then the real-valued functions defined for all  $r > 0$  by <sup>†</sup>

$$A_\lambda(r) = \frac{2\sqrt{2}}{d(\lambda)} \operatorname{Im}(J_1(\lambda\bar{\zeta})J_1(\lambda\zeta r)), \quad B_\lambda(r) = 0, \quad C_\lambda(r) = \frac{1}{d(\lambda)} \operatorname{Im}(\bar{\alpha}J_1(\lambda\bar{\zeta})J_0(\lambda\zeta r)) \quad (25)$$

are the unique solution of the differential system (17) satisfying the boundary conditions stated in Lemma 6.

*Proof.* Recall that, for  $\nu = 0, 1$ , Bessel's differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \quad (26)$$

---

<sup>†</sup>Note that the two determinations of the square root in the definition of  $\zeta$  yield the yield the same  $A_\lambda, B_\lambda$  and  $C_\lambda$ .

has a canonical basis of solutions consisting of the ordinary Bessel functions  $J_\nu(x)$  and  $Y_\nu(x)$  [11, §10.2]. The function  $J_\nu$  is entire, while  $Y_\nu$  is analytic on  $\mathbb{C} \setminus \mathbb{R}_-$ , and  $Y_\nu(x)$  diverges to  $-\infty$  as  $x \rightarrow 0$  along the positive reals.

Let  $\mathcal{C}_\nu$  denote a cylinder function, that is, a linear combination  $aJ_\nu + bY_\nu$  with coefficients  $a, b$  that do not depend on  $\nu$  [11, §10.2(ii)]. One has [11, (10.6.2), (10.6.3)]

$$\mathcal{C}'_0(x) = -\mathcal{C}'_1(x), \quad x\mathcal{C}'_1(x) + \mathcal{C}_1(x) = x\mathcal{C}_0(x). \quad (27)$$

The equation for  $B_\lambda$  is exactly Bessel's equation with  $\nu = 1$  and  $x = \lambda r$ ; its general solution for  $r > 0$  is hence  $B_\lambda(r) = \mathcal{C}_1(\lambda r)$ . Since  $Y_1(x)$  diverges as  $x \rightarrow 0$  and  $\lambda$  is nonzero, the initial condition at 0 forces  $b = 0$ . Similarly, the initial condition at 1 implies  $a = 0$  and therefore  $B_\lambda = 0$ , unless  $J_1(\lambda) = 0$ , in which case any  $B_\lambda(r) = aJ_1(\lambda r)$  is a solution.

Let us turn to the coupled equations for  $A_\lambda$  and  $C_\lambda$ . Make the ansatz

$$C_\lambda(r) = f_0(r), \quad A_\lambda(r) = \alpha f_1(r), \quad f_\nu(r) = \mathcal{C}_\nu(\lambda \zeta r), \quad (28)$$

where  $\alpha, \zeta$  are yet unspecified complex numbers. The equation involving  $C''_\lambda$  becomes

$$r f''_0(r) + f'_0(r) + 2\lambda^2 r f_0(r) + 2\lambda\alpha(r f'_1(r) + f_1(r)) = 0. \quad (29)$$

The change of variable passing from  $\mathcal{C}_0$  to  $f_0$  transforms Bessel's equation into

$$r f''_0(r) + f'_0(r) + \lambda^2 \zeta^2 r f_0(r) = 0,$$

and the relations (27) yield  $r f'_1(r) + f_1(r) = \lambda \zeta r f_0(r)$ , so (29) holds when

$$\lambda^2 \zeta^2 r = 2\lambda^2 r + 2\lambda\alpha \cdot \lambda \zeta r,$$

that is, when  $\zeta^2 = 2(1 + \alpha\zeta)$ .

Similarly, the equation involving  $A''_\lambda$  rewrites as

$$r^2 \alpha f''_1(r) + r \alpha f'_1(r) + (\lambda^2 r^2 - 1) \alpha f_1(r) + 2\lambda r^2 f'_0(r) = 0, \quad (30)$$

and the last term on the left-hand side is equal to  $2\lambda^2 \zeta r^2 f_1(r)$  by (27). Thus, (30) reduces to Bessel's equation provided that  $\zeta^2 = 1 - 2\alpha^{-1}\zeta$ .

In summary, the functions (28) define a solution of (17) for any choice of  $a, b$  in the definition of  $\mathcal{C}_\nu$  and  $\alpha, \zeta$  such that  $\zeta^2 = 2(1 + \alpha\zeta) = 1 - 2\alpha^{-1}\zeta$ . The latter condition is equivalent to

$$\zeta^4 + \zeta^2 + 2 = 0, \quad \alpha = \zeta^3/2 + \zeta.$$

Letting  $\zeta$  now denote a fixed root of  $\zeta^4 + \zeta^2 + 2$ , say the one in (24), the choices

$$C_\lambda(r) = J_0(\lambda \zeta r), J_0(\lambda \bar{\zeta} r), Y_0(\lambda \zeta r), Y_0(\lambda \bar{\zeta} r) \quad (31)$$

provide us with four linearly independent<sup>†</sup> solutions, which hence form a basis of the solution space of the system of two linear differential equation of order two.

The asymptotic behaviour of  $Y_1$  at the origin,  $Y_1(\lambda \zeta r) \sim -2(\pi \lambda \zeta r)^{-1}$  [11, (10.7.4)], shows that linear combinations involving any of the last two solutions (31) are incompatible with the conditions  $A_\lambda(0) = C'_\lambda(0) = 0$ . Therefore, one has

$$C_\lambda(r) = u J_0(\lambda \zeta r) + v J_0(\lambda \bar{\zeta} r),$$

$$A_\lambda(r) = u \alpha J_1(\lambda \zeta r) + v \bar{\alpha} J_1(\lambda \bar{\zeta} r)$$

for some  $u, v \in \mathbb{C}$ . The conditions  $A_\lambda(1) = 0, C_\lambda(1) = 1$  translate into a linear system for  $u, v$  of determinant

$$\bar{\alpha} J_0(\lambda \zeta) J_1(\lambda \bar{\zeta}) - \alpha J_1(\lambda \zeta) J_0(\lambda \bar{\zeta}) = 2id(\lambda)$$

(where we have used the fact that  $J_\nu(\bar{z}) = \overline{J_\nu(z)}$  [11, (10.11.9)]). When  $d(\lambda) \neq 0$ , the unique solution is  $u = -\bar{v} = \bar{\alpha} J_1(\lambda \bar{\zeta})$ . Since  $|\alpha|^2 = 2\sqrt{2}$ , this leads to the expressions (25).  $\square$

<sup>†</sup>This follows, for instance, from the expressions [11, (10.2.2), (10.8.1)] and the fact that  $(\lambda \zeta)^2 \neq 1$ .

## 6. Conclusion

LEMMA 8. *In the notation of Lemma 7, there exists  $\tilde{\lambda} > 0$  such that  $C(0)$ , viewed as a function of  $\lambda$ , has a pole at  $\tilde{\lambda}$ .*

*Proof.* Let us first show that  $d(\lambda)$  has a zero lying in the interval (2.5, 3). Consider the series expansions [11, (10.2.2)]

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k}}{k!^2}, \quad J_1(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+1}}{k!(k+1)!}. \quad (32)$$

For  $x \in \mathbb{C}$  and  $n \in \mathbb{N}$  such that  $|x| < 2(n+1)$ , the remainders starting at index  $n$  of both series are bounded by

$$\sum_{k=n}^{\infty} \frac{|x/2|^{2k}}{k!^2} = \frac{|x/2|^{2n}}{n!^2} \sum_{k=n}^{\infty} \frac{|x/2|^{2(k-n)}}{((n+1) \cdots (n+k))^2} \leq \frac{1}{1 - |x|^2/(2n+2)^2} \frac{|x/2|^{2n}}{n!^2}. \quad (33)$$

In particular, for  $x = \lambda\zeta$  or  $x = \lambda\bar{\zeta}$  with  $0 < \lambda \leq 3$ , we have  $|x/2| < 1.784$ . For  $n = 5$ , the quantity (33) is bounded by 0.025. By replacing  $J_0$  and  $J_1$  by the first five terms of the series (32) in the expression of  $d(\lambda)$  and propagating this bound by the triangle inequality, one can check that  $d(2.5) < -0.06$ . A similar calculation shows that  $d(3) > 0.03$ . Since  $d(\lambda)$  is a continuous function of  $\lambda$ , it follows that  $d(\tilde{\lambda})$  vanishes for some  $\tilde{\lambda} \in (2.5, 3)$ .

We still need to check that the numerator of  $C_\lambda(0)$  in (25) does not vanish at  $\tilde{\lambda}$ . One has  $J_0(0) = 1$ . Taking  $n = 3$  in (33) yields an expression of the form

$$\operatorname{Im}(\bar{\alpha}J_1(\lambda\bar{\zeta})) = \operatorname{Im}(c_0\lambda + c_1\lambda^3 + c_2\lambda^5 + \bar{\alpha}b), \quad |b| \leq 1.12,$$

where one can check that  $\operatorname{Im}(c_0) < -0.33$ ,  $\operatorname{Im}(c_1) < -0.12$ ,  $\operatorname{Im}(c_2) < -0.001$ . For all  $\lambda \geq 2.5$ , this implies

$$\operatorname{Im}(\bar{\alpha}J_1(\lambda\bar{\zeta})) \leq \operatorname{Im}(c_0)\lambda + \operatorname{Im}(c_1)\lambda^3 + |\alpha||b| \leq -1.3.$$

The claim follows.  $\square$

REMARK 9. Instead of doing the calculation sketched in the proof manually, one can easily prove the result using a computer implementation of Bessel functions that provides rigorous error bounds. For example, using the interval arithmetic library Arb [7] via SageMath, the check that  $d(\lambda)$  has a zero goes as follows. The quantities of the form `[x.xxx +/- eps]` appearing in the output are guaranteed to be rigorous enclosures of the corresponding real quantities. We check the presence of a zero in the interval [2.82, 2.83] instead of [2.5, 3.0] because having a tighter estimate simplifies the second step.

```
sage: zeta = CBF(sqrt((-1+I*sqrt(7))/2))
sage: alpha = zeta^3/2 + zeta
sage: lb, ub = CBF(282/100), CBF(283/100)
sage: (alpha.conjugate()*(lb*zeta).bessel_J(0)
.....:      *(lb*zeta.conjugate()).bessel_J(1))
[-13.208370024264 +/- 4.16e-13] + [-0.003639973760 +/- 4.63e-13]*I
sage: (alpha.conjugate()*(ub*zeta).bessel_J(0)
.....:      *(ub*zeta.conjugate()).bessel_J(1))
[-13.424373315124 +/- 4.75e-13] + [0.005782411521 +/- 4.38e-13]*I
```

One can then verify as follows that the image by the function  $\lambda \mapsto \bar{\alpha}J_1(\lambda\bar{\zeta})$  of the interval [2.82, 2.83] only contains elements of negative imaginary part.

```
sage: crit = lb.union(ub); crit # convex hull (real interval)
[2.8 +/- 0.0301]
sage: alpha.conjugate()*(crit*zeta.conjugate()).bessel_J(1)
[+/- 0.0707] + [-4e+0 +/- 0.303]*I
```

THEOREM 10. *The series expansion with respect to  $\lambda$  of  $\Phi(0)$  has a finite radius of convergence.*

REMARK 11. Since the condition of [2] of uniqueness of laws is only *sufficient*, the question remains on whether there exists another law on  $G$  having the same moments as  $S(X^0)_{0,T}$ .

*Proof.* Assume for contradiction that  $\Phi(0)$  has an infinite radius of convergence. Then  $F_\lambda(0)$  is an entire function in  $\lambda$ . We also know from Corollary 5 that there exists  $\lambda^* > 0$  such that for real  $\lambda < \lambda^*$

$$F_\lambda(0) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\lambda(0) \\ B_\lambda(0) \\ C_\lambda(0) \end{pmatrix}, \quad (34)$$

where  $A_\lambda, B_\lambda, C_\lambda$  are defined by Lemma 7. By the Identity theorem,  $A_\lambda, B_\lambda, C_\lambda$  are entire functions and (34) holds for all  $\lambda$ . This contradicts Lemma 8, and therefore  $\Phi(0)$  has a finite radius of convergence.  $\square$

### Appendix

Let  $\Gamma$  be a domain in  $\mathbb{R}^d$ .

DEFINITION A.1. Let  $u$  be a locally integrable function in  $\Gamma$  and  $\alpha$  be a multi-index. Then a locally integrable function  $r_\alpha u$  such that for every  $g \in C_c^\infty(\Gamma)$ ,

$$\int_\Gamma g(x) r_\alpha(x) dx = (-1)^{|\alpha|} \int_\Gamma D^\alpha g(x) u(x) dx,$$

will be called *weak derivative* of  $u$  and  $r_\alpha$  is denoted by  $D^\alpha u$ . By convention,  $D^\alpha u = u$  if  $|\alpha| = 0$ .

DEFINITION A.2. Let  $\tilde{d} \in \mathbb{N}$ . The *Sobolev space*  $W^{k,p}(\Gamma)$  for  $p, k \in \mathbb{N}$  is defined to be the set of all  $\mathbb{R}^{\tilde{d}}$ -valued functions  $u = (u^1, \dots, u^{\tilde{d}}) \in L^p(\Gamma)$  such that for every multi-index  $\alpha$  with  $|\alpha| \leq k$ , the weak partial derivative  $D^\alpha u$  belongs to  $L^p(\Gamma)$ , that is,

$$W^{k,p}(\Gamma) = \{u \in L^p(\Gamma) : D^\alpha u \in L^p(\Gamma) \forall |\alpha| \leq k\}.$$

It is endowed with the Sobolev norm defined as follows:

$$\|u\|_{W^{k,p}(\Gamma)} = \sum_{j=1}^{\tilde{d}} \left( \sum_{|\alpha| \leq k} \int_\Gamma |D^\alpha u^j(x)|^p dx \right)^{1/p}.$$

When  $k = 0$ , this norm coincides with the  $L^p(\Gamma)$ -norm, that is,

$$\|u\|_{W^{0,p}(\Gamma)} = \|u\|_{L^p(\Gamma)}.$$

THEOREM A.3. *Let  $M$  be a second order differential operator with coefficients  $\{a^{i,j}\}$ . Let  $u$  be a weak solution of*

$$\begin{aligned} Mu &= f(x), \\ u - g &\in H_0^{1,2}(\Gamma). \end{aligned}$$

Let  $\Gamma$  be a bounded domain of class  $C^{k+2}$  and let the coefficients of  $M$  be of class  $C^{k+1}(\bar{\Gamma})$ . Suppose that the following ellipticity condition holds: there exists  $\lambda > 0$  such that for all  $x \in \Gamma$  and all  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a^{i,j}(x) \xi_i \xi_j \geq \lambda |\xi|^2.$$

Let  $f \in W^{k,2}(\Gamma)$ ,  $g \in W^{k+2,2}(\Gamma)$ . Then

$$\|u\|_{W^{k+2,2}(\Gamma)} \leq c(\|f\|_{W^{k,2}(\Gamma)} + \|g\|_{W^{k+2,2}(\Gamma)}),$$

with  $c$  depending on  $\lambda, d, \Gamma$  and on the  $C^{k+1}$ -norms for the  $a^{i,j}$ .

*Proof.* It is proved using [5, Theorem 8.13] and setting the boundary condition  $\varphi = 0$ .  $\square$

In the following we prove Lemma A.4 for  $m \geq 2$ , which is a generalization of Lemma 3.11 for the case  $m = \lfloor \frac{d}{2} \rfloor$  in [8].

LEMMA A.4. *Let  $\Gamma$  be a bounded domain of class  $C^m$  in  $\mathbb{R}^d$ , where  $m \geq 2$ . Then there exists a constant  $C$  only depending on  $\Gamma$  and  $d$ , such that for every positive integer  $n \geq 2$ ,*

$$\|\text{proj}_n(\Phi)\|_{W^{m,2}(\Gamma)} \leq C(\|\text{proj}_{n-1}(\Phi)\|_{W^{m,2}(\Gamma)} + \|\text{proj}_{n-2}(\Phi)\|_{W^{m,2}(\Gamma)}). \quad (\text{A.1})$$

*Proof.* The proof of [8, Lemma 3.11] can be applied here directly, except for that we need to check that  $\text{proj}_n(\Phi) \in W^{m,2}$ , which is proved in Theorem A.5.  $\square$

THEOREM A.5. *Suppose that  $\Gamma$  is a nonempty bounded domain in  $E$ . It follows  $\Phi$  is infinitely differentiable in componentwise sense, that is, for all index  $I$ ,  $\text{proj}_n \circ \Phi$  is infinitely differentiable for all  $n$ .*

*Proof.* Based on [8, Theorem 3.2], it shows that

$$\Phi(z) = \int_{\Gamma} G_{\varepsilon}(z-y) \otimes \Phi_{\Gamma}(y) dy = G_{\varepsilon} * \Phi(z),$$

where  $K_{\varepsilon}(r)$  is a smooth distribution with compact support  $[0, \frac{\varepsilon}{2}]$ ,  $*$  is the convolution, and  $G_{\varepsilon}$  be a map from  $\mathbb{R}^d$  to  $T(\mathbb{R}^d)$  defined by

$$\Psi(z) = \Psi(z) = \mathbb{E}^0 \left[ S(B_{[0, \tau_{\mathbb{D}(0,|z|)}]}) | B_{\tau_{\mathbb{D}(0,|z|)}} = z \right].$$

$$G_{\varepsilon}(z) = \Psi(-z) K_{\varepsilon}(|z|).$$

Since  $\Psi$  is smooth (in polynomial form) and  $K_{\varepsilon}$  is a smooth function with compact support,  $G_{\varepsilon}$  is a smooth function with compact support. It is easy to show that for any partial derivative  $D^{\alpha} G_{\varepsilon}$  is  $L_1$  integrable.

$$\|D^{\alpha} G_{\varepsilon}\|_{L^1} < +\infty.$$

On the other hand,  $\Phi \in L^1$  as well, and so we have

$$\|G_{\varepsilon} * D^{\alpha} \Phi\|_{L^1} < +\infty.$$

Thus  $G_{\varepsilon} * \Phi$  is infinitely differentiable, since  $D^{\alpha} \Phi = (D^{\alpha} G_{\varepsilon}) * \Phi \in L^1$ .  $\square$

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