

Numerical Evaluation of D-Finite Functions in SageMath with Applications

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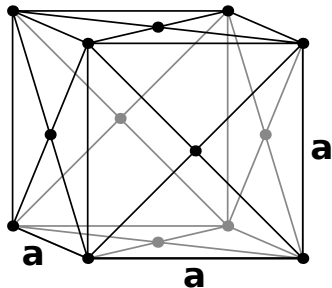
Workshop on Computer Algebra in Combinatorics
Erwin Schrödinger Institute, Wien, 2017-11-16

[arxiv:1607.01967](https://arxiv.org/abs/1607.01967) [cs.SC]

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Face-Centered Cubic Lattices

[Guttman 2009, Broadhurst 2009, Koutschan 2013]



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$$\text{return probability} = 1 - \frac{1}{P(1)}$$

dimension d :

$$P(z) = \frac{1}{\pi^d} \int_0^\pi \dots \int_0^\pi \frac{dk_1 \dots dk_d}{1 - \lambda(\mathbf{k})z}$$

$$\lambda(\mathbf{k}) = \frac{2}{d(d-1)} \sum_{i < j} \cos k_i \cos k_j$$

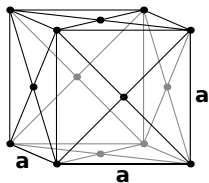
For fixed d :

- ▶ Compute a **differential operator** \mathcal{D} such that $\mathcal{D} \cdot P = 0$
- ▶ Compute $P(1)$ by solving the differential equation

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} \text{dop6} = & 410085196915322880 z^{35} + 112905266474211563520 z^{34} + 1171669263761496 \backslash \\ & 1489920 z^{33} + 690817401287078917363200 z^{32} + 27204862643846611522761600 z^{31} + 7788 \backslash \\ & 11406918247228618497600 z^{30} + 17044384124115240781429792800 z^{29} + 29424523406685 \backslash \\ & 0000428339092800 z^{28} + 4083424587805117060272476125800 z^{27} + 4597302954911979623 \backslash \\ & 5386142827300 z^{26} + 419695598890898253203455876749930 z^{25} + 30642971761740916717 \backslash \\ & 17985958725620 z^{24} + 17169584489259696388755804636033570 z^{23} + 64581771961684810 \backslash \\ & 077279475394020500 z^{22} + 51714221934272099420476126216766700 z^{21} - 1473967391504 \backslash \\ & 437899277380487903179960 z^{20} - 14237554341321335335392023192872385940 z^{19} - 8321 \backslash \\ & 6340134393115016834220980384454340 z^{18} - 36401915432810756256884790682248806 \backslash \\ & 3550 z^{17} - 1261571478513401088177035093275526304300 z^{16} - 3528341032098896995323 \backslash \\ & 439017117956856150 z^{15} - 7964369518593778029521056070442794466900 z^{14} - 14280500 \backslash \\ & 726162786254712841163875001728600 z^{13} - 1953465311568634254358083196094197891 \backslash \\ & 8000 z^{12} - 18398783334222380084238012428704731960000 z^{11} - 755374178599030935723 \backslash \\ & 4054786177488000000 z^{10} + 8887432309419522403983976171775697600000 z^9 + 21137039 \backslash \\ & 158366320685856256980012112000000 z^8 + 2268269355393480 \backslash \\ & 46904466472955088000000000 z^7 + 149381834281462611905463546 \backslash \\ & 71616000000000 z^6 + 469024652858481632994019940044800000 \backslash \\ & 0000 z^5 - 872829008634785738926162452480000000000 z^4 - 11043 \backslash \\ & 27940779745890150773145600000000000 z^3 - 35389870820758085 \backslash \\ & 67729192960000000000000 z^2 - 520793429107744448741376000000 \backslash \\ & 0000000 z - 242879062193188503552000000000000000 + \end{aligned}$$



Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (3964156903514787840 z^{36} + 1104718489963413534720 z^{35} + \\ & 117871088739930352834560 z^{34} + 7183287516644479615795200 z^{33} + 29310583 \backslash \\ & 5218942903781855360 z^{32} + 8706572378734984776799502400 z^{31} + 1979497761381158661 \backslash \\ & 33849254880 z^{30} + 3555494624146318548046453851120 z^{29} + 514578986720138650981112 \backslash \\ & 91247320 z^{28} + 606522834979531840684521625237020 z^{27} + 5835366836846027182876920 \backslash \\ & 856348950 z^{26} + 45455202501826358974606219981974015 z^{25} + 2791534044675020629485 \backslash \\ & 31557838260750 z^{24} + 1252275399372837134061507042628908795 z^{23} + 294318280292355 \backslash \\ & 2038161307584706940070 z^{22} - 10483513115206289398510413216920199750 z^{21} - 169948 \backslash \\ & 182933507479161257565568616530700 z^{20} - 115496959477627716064907778598382055 \backslash \\ & 3870 z^{19} - 5548694490781020038019823355124585193590 z^{18} - 2074522951757745127237 \backslash \\ & 7158241970915439245 z^{17} - 62232963928794638659423069651761724690290 z^{16} - 150810 \backslash \\ & 045901978932864163493046405461262105 z^{15} - 292528626523005661629390236883046859 \backslash \\ & 976150 z^{14} - 441395096063183148839008172248580337780300 z^{13} - 484123578764537043 \backslash \\ & 031861206473715269343000 z^{12} - 312976584649334763451810663858004196420000 z^{11} + \\ & 31415133499909950234831915395869293600000 z^{10} + 330738460087674555468491482629 \backslash \\ & 558468000000 z^9 + 391096978918364972225128472061480072000000 z^8 + 23246017042594 \backslash \\ & 8027345434850305279520000000 z^7 + 18060134934884299834847099345568000000000 z^6 - \\ & 100213400891192102370293326036992000000000 z^5 - 8385906451598513649590309945856 \backslash \\ & 0000000000 z^4 - 29487051768804049058487328972800000000000 z^3 - 55164827197430422 \backslash \\ & 28472840192000000000000 z^2 - 37955378966350598718455808000000000000 z + 8379327 \backslash \\ & 645665003372544000000000000000) Dz + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (8133356405487237120 z^{37} + 2294131782043664317440 z^{36} + \\ & 251295328534762193633280 z^{35} + 15795453015240816970091520 z^{34} + 666093618246 \backslash \\ & 765502077439680 z^{33} + 20469375712416843040909376160 z^{32} + 4818173382673386397833 \backslash \\ & 28749120 z^{31} + 8967973126212020032517991216960 z^{30} + 134696914854304536722281866 \backslash \\ & 954300 z^{29} + 1651833654984876079820125885678650 z^{28} + 16607490026343429532811575 \backslash \\ & 311949230 z^{27} + 136263869454304799146859253346813455 z^{26} + 895865319327471447638 \backslash \\ & 111289873238710 z^{25} + 4489711074264384906529925990254793265 z^{24} + 14491852283494 \backslash \\ & 577826654003932547711690 z^{23} - 168509066471194546983174648133542750 z^{22} - 386265 \backslash \\ & 894549826881229123104470731096440 z^{21} - 316325913106056858454611334378198756 \backslash \\ & 1220 z^{20} - 16636182069413821170544684047556220568150 z^{19} - 662467400893936760809 \backslash \\ & 81537130378090658525 z^{18} - 209080245872850631566312137449619561543730 z^{17} - \\ & 529097465740104776391772834675033946593335 z^{16} - 10650386207573139293916393610 \backslash \\ & 32363453750930 z^{15} - 1653651644685620142167009422124022555221700 z^{14} - 182938347 \backslash \\ & 4513975929874027770563298831967800 z^{13} - 10887098968369058066602841492776215683 \backslash \\ & 28000 z^{12} + 437384067337328886944483963336952904080000 z^{11} + 1678380365365432006 \backslash \\ & 625451473236269012000000 z^{10} + 1564385355592027935922683162898655112000000 z^9 + \\ & 303607398715325954207032303663107840000000 z^8 - 929554263384554771136483801 \backslash \\ & 584745600000000 z^7 - 133465853572648237153690804917964800000000 z^6 - 9419775340 \backslash \\ & 0652483718287926356480000000000 z^5 - 35029765385278767758927501660160000000 \backslash \\ & 0000 z^4 - 6428624147389214823464058470400000000000 z^3 - 37458505751548575098585 \backslash \\ & 08800000000000000 z^2 + 66870677798339624740454400000000000000 z + 7286371865795 \backslash \\ & 6551065600000000000000000) Dz^2 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (6219625486549063680 z^{38} + 1775531336308022522880 z^{37} + \\ & 199409996635132589752320 z^{36} + 12904862497592448920163840 z^{35} + 561222248755 \backslash \\ & 128125708191680 z^{34} + 17798695421072697669468739680 z^{33} + 4325301686047256581892 \backslash \\ & 10596640 z^{32} + 8315189920333341531658617695280 z^{31} + 129103723904595771409928232 \backslash \\ & 487740 z^{30} + 1639190738531986170699647097803790 z^{29} + 17111040709840823035760757 \backslash \\ & 618682440 z^{28} + 146518901443861803658771329866897880 z^{27} + 101553427880666984315 \backslash \\ & 9745327151252620 z^{26} + 5496338276053076075068754467310102760 z^{25} + 2089057420971 \backslash \\ & 4927539267068315744951640 z^{24} + 30164609970591947189827076234922007050 z^{23} - \\ & 289127416281529376142095631015519267120 z^{22} - 30536692798730633467931505919379 \backslash \\ & 74700130 z^{21} - 17566486109105161467894504789161406270600 z^{20} - 73673650638461574 \backslash \\ & 538679743097050051115220 z^{19} - 240407438967557398913317296975336574702980 z^{18} - \\ & 619168293687639511251067273975020197114100 z^{17} - 12412253032904606237959908599 \backslash \\ & 05959226579320 z^{16} - 1835553795134837646262350779261931882894750 z^{15} - 167031460 \backslash \\ & 2837141110845640706031012073555700 z^{14} + 3066874861862296186275087607527571093 \backslash \\ & 2000 z^{13} + 2931594155313390328935716187001614568260000 z^{12} + 4919456458899666498 \backslash \\ & 684069708388548285600000 z^{11} + 3777365646243762653104795884206143332000000 z^{10} + \\ & 68195639154415674514017863641593600000000 z^9 - 31188109375232537262356667826 \backslash \\ & 66096800000000 z^8 - 3771833787399616704258908808294288000000000 z^7 - 24884448319 \backslash \\ & 30996824908954989144320000000000 z^6 - 89395910342209380332330526274560000000 \backslash \\ & 0000 z^5 - 100083705719332806676962561024000000000000 z^4 + 2564203084567737476418 \backslash \\ & 01728000000000000000 z^3 + 1028076103183337304001413120000000000000 z^2 + 10337540 \backslash \\ & 08459758568243200000000000000000 z) Dz^3 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (2192816677949990400 z^{39} + 633490213477308768000 z^{38} + \\ & 72864986011484455353600 z^{37} + 4847486869795537260532800 z^{36} + 21701401 \backslash \\ & 7048761645614816000 z^{35} + 7088016995800124707996090560 z^{34} + 1774096571316102704 \backslash \\ & 82016190640 z^{33} + 3513089912736584156549238676620 z^{32} + 562015877327404499596706 \backslash \\ & 75451690 z^{31} + 735847759326730529024504240988015 z^{30} + 7934411063073314432988482 \backslash \\ & 485900500 z^{29} + 70405171912630286359945571896774110 z^{28} + 5088828139208506106992 \backslash \\ & 35633677324220 z^{27} + 2913386772290646501282812655546011475 z^{26} + 122379897749644 \backslash \\ & 63385062890926963215950 z^{25} + 27303163874555616052155898475524386210 z^{24} - 84400 \backslash \\ & 724272601405065271773264397209530 z^{23} - 130932954808576856297312907253772416 \backslash \\ & 4955 z^{22} - 8229269199062442444264260234977847805360 z^{21} - 3594874091884447514031 \backslash \\ & 8574840001115213670 z^{20} - 119246681320333134593914488403328142970080 z^{19} - 30427 \backslash \\ & 3308297438630162837099041285050546455 z^{18} - 57714503089590179069731112638689603 \backslash \\ & 6767490 z^{17} - 707778167790136602728038144967670916837350 z^{16} - 15334140655390733 \backslash \\ & 4245470038125935935813900 z^{15} + 1611202920628825942940219406515876419542000 z^{14} + \\ & 4188993616205017046899739124211544543460000 z^{13} + 5699626392082018037453259396 \backslash \\ & 194906388000000 z^{12} + 4145140187203309836183311398252469964000000 z^{11} + 95068892 \backslash \\ & 397133773199630362250506960000000 z^{10} - 322447694713681036240924354097202960000 \backslash \\ & 0000 z^9 - 36368355281389283027676649873995360000000000 z^8 - 210391871123659328537 \backslash \\ & 31961115328000000000000 z^7 - 55159060242541438416411785825280000000000 z^6 + 1162 \backslash \\ & 15925694410902420898178048000000000000 z^5 + 11283852768448237101798156288000000 \backslash \\ & 000000 z^4 + 26640440911690813973554790400000000000000 z^3 + 24272726277931776073 \backslash \\ & 72800000000000000000 z^2) Dz^4 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (390720062616543744 z^{40} + 114216661424360307456 z^{39} + \\ & 13440822351615963069696 z^{38} + 917965180366474611870720 z^{37} + 422376739322 \backslash \\ & 63775988570560 z^{36} + 1418218839310481932976078400 z^{35} + 364872068364080011976899 \backslash \\ & 10640 z^{34} + 742504152062765602237759452720 z^{33} + 1220569466691901164246256065 \backslash \\ & 0930 z^{32} + 164251022394443778986763736539405 z^{31} + 18217671156128364340537374047 \backslash \\ & 55054 z^{30} + 16657938302824007267243724365434191 z^{29} + 12453346084962020000971144 \backslash \\ & 5328730256 z^{28} + 743593796442908540070532245488378205 z^{27} + 33360046070881075316 \backslash \\ & 34361889061221370 z^{26} + 9020222161854736084202629824390547047 z^{25} - 919882472294 \backslash \\ & 3404205447421299404277112 z^{24} - 279449868802402514175677041789492570017 z^{23} - \\ & 1907863427661939885576723126598906643790 z^{22} - 8574083646475710050757565542672 \backslash \\ & 979674555 z^{21} - 28405587296847231070183606856583770811720 z^{20} - 6957425817531295 \backslash \\ & 5514440713973653616428745 z^{19} - 114991436892487711669937849824912517430330 z^{18} - \\ & 70378017201579863364495432167182725333675 z^{17} + 261641966501089147843656083216 \backslash \\ & 157842879550 z^{16} + 1049410824795136384837467209810025539400000 z^{15} + 20896637809 \backslash \\ & 64272997600159898811800513390000 z^{14} + 259786002679636380331331345092974504000 \backslash \\ & 0000 z^{13} + 1759136834585156085432113720072647266000000 z^{12} - 1346756149912237131 \backslash \\ & 08740928290811280000000 z^{11} - 1578996098791370746284707453439169200000000 z^{10} - \\ & 1556811681322720025894531955998040000000000 z^9 - 67890891761349944176134243479 \backslash \\ & 5200000000000 z^8 + 3149221592935046990103881875200000000000 z^7 + 192407344459752 \backslash \\ & 4252611218334720000000000000 z^6 + 10412684021044478206062895104000000000000 z^5 + \\ & 20982276643045393599012864000000000000000 z^4 + 17874380983279966433280000000000 \backslash \\ & 00000000 z^3) Dz^5 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (35882454730090752 z^{41} + 10612604051614486656 z^{40} + 1276532600942212775168 z^{39} + \\ & 89393980129433032096320 z^{38} + 4221606838983473228197008 z^{37} + 145494567985766 \backslash \\ & 484898923048 z^{36} + 3840828004490920060950969480 z^{35} + 8016006238826772717221198 \backslash \\ & 5080 z^{34} + 1350855094398006902682870922050 z^{33} + 1863108289263053682422294940 \backslash \\ & 9585 z^{32} + 211815796834464054711973645322142 z^{31} + 19867083220856675726655250160 \backslash \\ & 37411 z^{30} + 15263082383031406770429022758762048 z^{29} + 94068732852089205756130773 \backslash \\ & 605094705 z^{28} + 441055376229095921513357130918811338 z^{27} + 131963694549876126497 \backslash \\ & 3744224282378779 z^{26} - 137626809673226795399591264079041112 z^{25} - 31072001737970 \backslash \\ & 299221405533198706303141 z^{24} - 226886176666918560987240200768631693150 z^{23} - \\ & 1033954017266382248984767586852072344191 z^{22} - 335673294622437360164908793734 \backslash \\ & 9109785896 z^{21} - 7573126212785007618891225542456994124245 z^{20} - 9076459539413303 \backslash \\ & 184641722134776573895810 z^{19} + 10278671248090335377408918358815408788425 z^{18} + \\ & 85149274357043292385925033653294291853550 z^{17} + 2406893603584982960079390961 \backslash \\ & 87740586134000 z^{16} + 429409878921957648790555775268242743350000 z^{15} + 4957792250 \backslash \\ & 46771906420255540348281344800000 z^{14} + 28712136337931261687156234648446537800 \backslash \\ & 0000 z^{13} - 119682652007548350954457856750250720000000 z^{12} - 39568346559268086740 \backslash \\ & 1293480616198000000000 z^{11} - 32738346275504238594974769124082400000000 z^{10} - \\ & 86642575450501391066787202019520000000000 z^9 + 59704683972170679548931977222400 \backslash \\ & 000000000 z^8 + 7251161027741239099083936307200000000000 z^7 + 338828967558720719 \backslash \\ & 56886261760000000000000 z^6 + 631115677130491732576665600000000000000 z^5 + 51232 \backslash \\ & 3021813756999680000000000000000000 z^4) Dz^6 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (1600200173148416 z^{42} + 478782978712278912 z^{41} + 58815380786135567104 z^{40} + \\ & 4218590040421804170816 z^{39} + 204216444469816446653424 z^{38} + 7214118 \backslash \\ & 624119937529541160 z^{37} + 195106070712453547506798168 z^{36} + 416887031952436819753 \backslash \\ & 3959000 z^{35} + 71874412795312940511795668940 z^{34} + 101352803991324920736784237 \backslash \\ & 8270 z^{33} + 11775924181048893848357395670676 z^{32} + 11286205581821339235627976822 \backslash \\ & 5402 z^{31} + 886340494836475569866741139358344 z^{30} + 55926759731865674372797336853 \backslash \\ & 51646 z^{29} + 26983635759333711243427828354079724 z^{28} + 85059388463264142313662526 \backslash \\ & 542420618 z^{27} + 24816956833181644480403313150735864 z^{26} - 1739731529923503295984 \backslash \\ & 796806526752758 z^{25} - 13215685421423157401833903137021991092 z^{24} - 6010151473251 \backslash \\ & 7779329542749898893453858 z^{23} - 187406121933740017212741167478185137320 z^{22} - \\ & 367088786736715063908412462166156515566 z^{21} - 136331238303988349001415414181 \backslash \\ & 532146340 z^{20} + 2052937632229799753666758504303681446150 z^{19} + 89422208647113020 \backslash \\ & 92023950168348534856300 z^{18} + 22112779083456047399791690319673356808000 z^{17} + \\ & 36662299830964853548300895468723502480000 z^{16} + 3866393620905473995570164907678 \backslash \\ & 4708400000 z^{15} + 15575841209632684184725074680551176000000 z^{14} - 233997759271107 \backslash \\ & 78754739301057544560000000 z^{13} - 4595758184455506810833869296180720000000 z^{12} - \\ & 32525005285459811112066289505232000000000 z^{11} - 3661218292337523929046664304 \backslash \\ & 640000000000 z^{10} + 1097039530150661129281453716480000000000 z^9 + 97131124051979 \backslash \\ & 35942595533824000000000000 z^8 + 426097871942078389250377728000000000000 z^7 + \\ & 75646707912253502668800000000000000000 z^6 + 592017714095896977408000000000000 \backslash \\ & 0000 z^5) Dz^7 + \end{aligned}$$

Face-Centered Cubic Lattices ($d = 6$)

[Koutschan 2013]

$$\begin{aligned} & (27122036833024 z^{43} + 8208413201024064 z^{42} + 1028987679702510976 z^{41} + 7551845 \backslash \\ & 1137118783792 z^{40} + 3743195619381989907184 z^{39} + 135369638077546936261428 z^{38} + \\ & 3745615314367420203992832 z^{37} + 81811619367860049045984675 z^{36} + 1440466637248 \backslash \\ & 203913774334250 z^{35} + 20724331113040275023719172850 z^{34} + 2454466275416520460977 \backslash \\ & 92768214 z^{33} + 2395828801191215780780578117794 z^{32} + 191474074706731112318622494 \backslash \\ & 18166 z^{31} + 122863963621496746370188659696702 z^{30} + 6026212556484859243787006723 \backslash \\ & 31054 z^{29} + 1935192664301617476137337671088360 z^{28} + 694152712036783264243644290 \backslash \\ & 673234 z^{27} - 39030042885818935455901289133872622 z^{26} - 2976459628039331965648737 \backslash \\ & 33670191774 z^{25} - 1329742929728007215704002549281591538 z^{24} - 390398961482564881 \backslash \\ & 9224432657208727646 z^{23} - 6038015534019664017777438417359311914 z^{22} + 7565280951 \backslash \\ & 156009750992823479550694170 z^{21} + 83328126336960183101771239549883786325 z^{20} + \\ & 297859836972471180382017327162905955900 z^{19} + 681226694393685252017130073908 \backslash \\ & 325840500 z^{18} + 1055504316932586226613390044310017920000 z^{17} + 97498262514411065 \backslash \\ & 4834660688990434600000 z^{16} + 80921481727424794623135930623472000000 z^{15} - 124569 \backslash \\ & 2778975371208980497936649580000000 z^{14} - 187761297216604654284152589154800000 \backslash \\ & 0000 z^{13} - 1186201691981014544058180007080000000000 z^{12} - 4424139263701398029187 \backslash \\ & 830400000000000 z^{11} + 52658849885502355913417731200000000000 z^{10} + 410999234738 \backslash \\ & 8340102474695680000000000000 z^9 + 17433301221381095805118464000000000000 z^8 + \\ & 29950530354743793153638400000000000000 z^7 + 22769912080611422208000000000000 \backslash \\ & 0000 z^6) Dz^8 \end{aligned}$$

order 8, degree 43, 43-digit coefficients

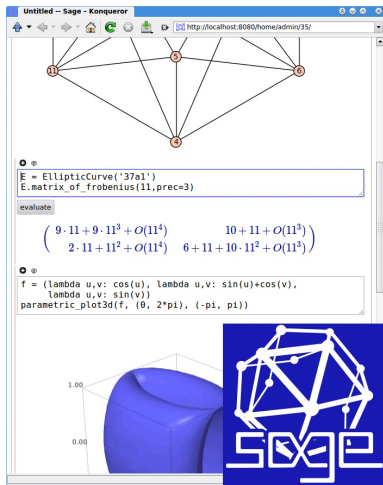
Walks with Small Steps in the Quarter Plane

[Bostan & Kauers '09; Melczer & Wilson '16; Bostan, Chyzak, van Hoeij, Kauers & Pech '17]

	OEIS	\mathfrak{S}	alg	asympt		OEIS	\mathfrak{S}	alg	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
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$$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

SageMath



The screenshot shows the SageMath web interface in a browser window titled "Untitled - Sage - Konqueror". The address bar shows "http://localhost:8080/home/admin/35/". The main content area displays a graph with five nodes and several edges. Below the graph, there is a code editor with the following code:

```
E = EllipticCurve('37a1')
E.matrix_of_frobenius(11, prec=3)
```

An "evaluate" button is visible below the code. The output shows a matrix of Frobenius elements:

$$\begin{pmatrix} 9 \cdot 11 + 9 \cdot 11^3 + O(11^4) & 10 + 11 + O(11^3) \\ 2 \cdot 11 + 11^2 + O(11^4) & 6 + 11 + 10 \cdot 11^2 + O(11^3) \end{pmatrix}$$

Below the matrix, there is another code editor with the following code:

```
f = (lambda u,v: cos(u), lambda u,v: sin(u)+cos(v),
     lambda u,v: sin(v))
parametric_plot3d(f, (0, 2*pi), (-pi, pi))
```

The bottom part of the screenshot shows a 3D plot of a blue surface and the SageMath logo.

- ▶ Python library
- ▶ “A viable alternative to Magma, Maple, Mathematica and Matlab”
- ▶ Based on many pre-existing systems and libraries



<http://sagemath.org/>

GNU GPL v2+



<http://cocalc.com/>



<http://ask.sagemath.org/>

Computational Mathematics with SageMath



A. Casamayou • N. Cohen • G. Connan •
T. Dumont • L. Fousse • F. Maltey • M. Meulien
• M. Mezzarobba • C. Pernet • N. M. Thiéry
• P. Zimmermann

English translation by

E. Bray • G. Connan • J. Cremona • T. Dumont •
M. Forets • A. Ghitza • C. Pernet • N. M. Thiéry
• H. Thomas • P. Zimmermann

coming soon!



<http://sagebook.gforge.inria.fr/>

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```
sage: from ore_algebra import OreAlgebra
sage: DiffOps.<Dz> = OreAlgebra(Pols)
sage: DiffOps
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

This talk: **Numerical analytic continuation** of solutions of **univariate differential operators**



<http://kauers.de/software.html>

GNU GPL v2+



http://marc.mezzarobba.net/code/ore_algebra-analytic/

```
sage: from ore_algebra import OreAlgebra
```

```
sage: DiffOps.<Dz> = OreAlgebra(Pols)
```

```
sage: DiffOps
```

```
Univariate Ore algebra in Dz over Univariate Polynomial Ring in z  
over Rational Field
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

This talk: **Numerical analytic continuation** of solutions of **univariate differential operators**



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GNU GPL v2+



http://marc.mezzarobba.net/code/ore_algebra-analytic/

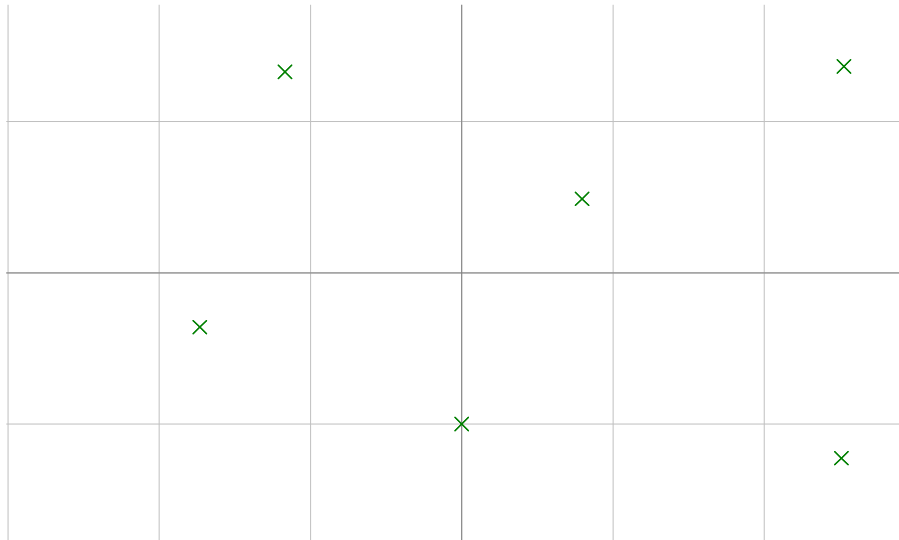
Analytic Continuation

$$\mathcal{L} = a_r(z) Dz^r + \cdots + a_1(z) Dz + a_0$$



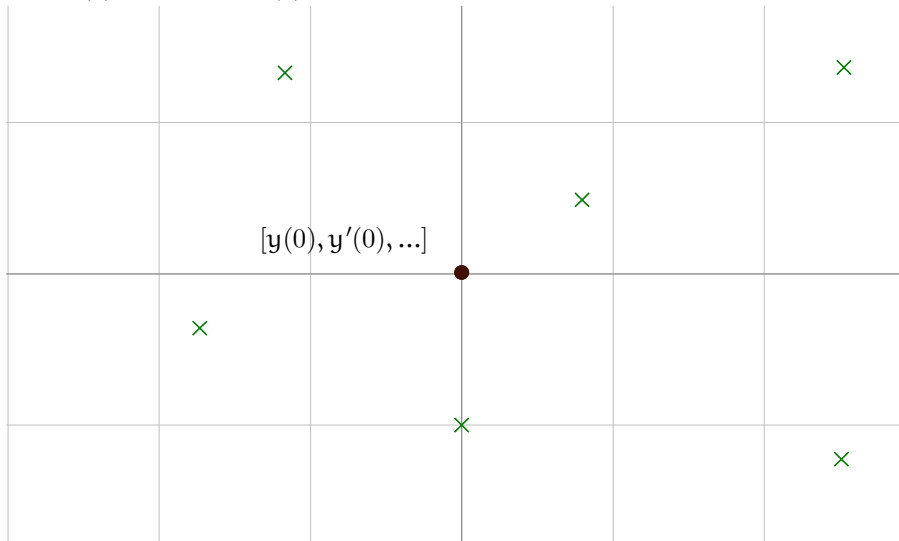
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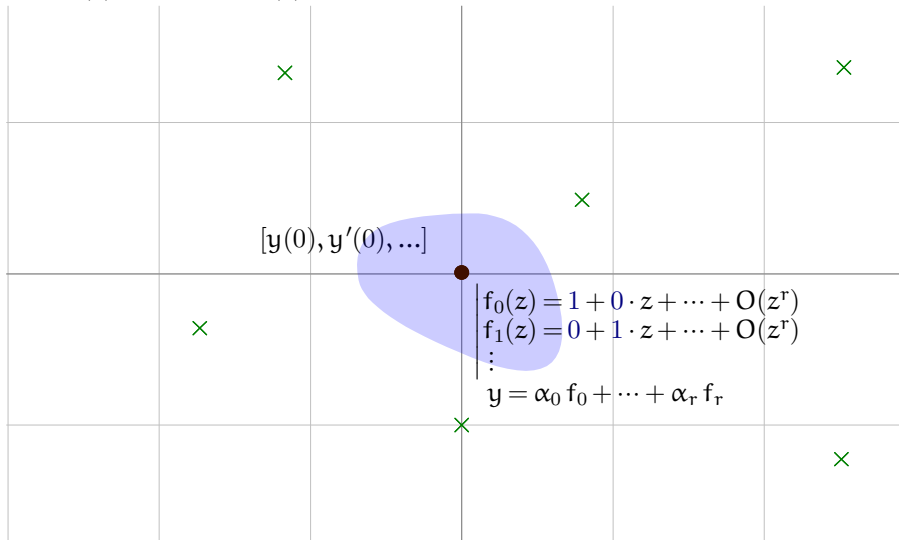
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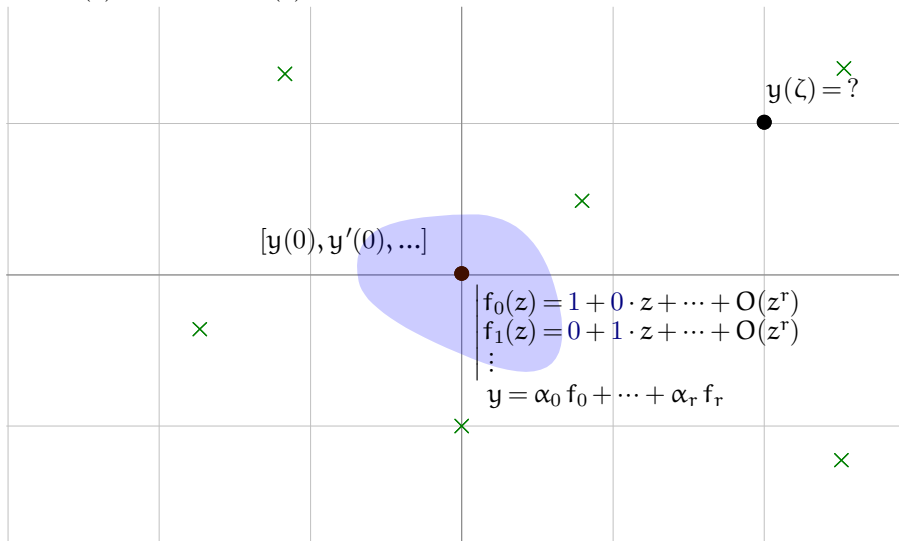
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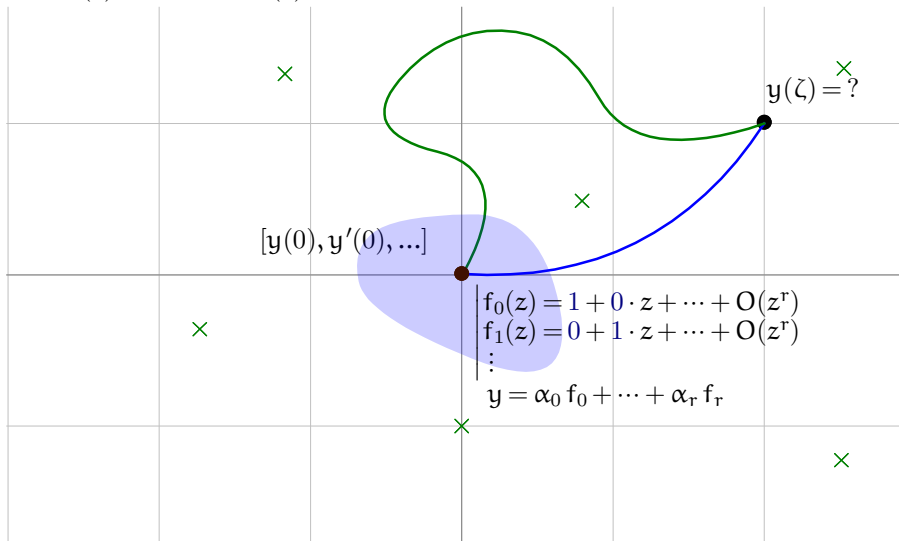
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Analytic Continuation

$$\mathcal{L} = a_r(z) Dz^r + \dots + a_1(z) Dz + a_0$$



Regular Singular Points

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0 \quad (\text{Bessel eq.})$$

 **regular singular point** at 0

✓ $z^{-3/2} \log z$

✓ $z^{i\sqrt{2}}$

✗ $e^{\pm 1/z}$

Theorem

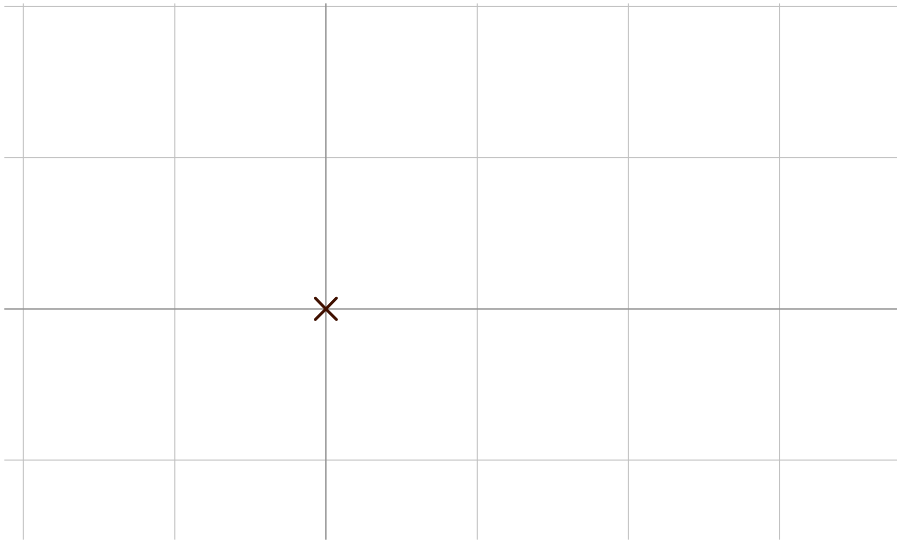
[Fuchs, 1866]

Assume that 0 is a regular singular point. Then, for some neighborhood D of 0, there exists a basis of solutions defined on $D \setminus \{0\}$ of the form

$$z^\lambda (y_0(z) + y_1(z) \log z + \cdots + y_t(z) \log^t z), \quad \lambda \in \bar{\mathbb{Q}}, \quad y_i \text{ analytic on } D.$$

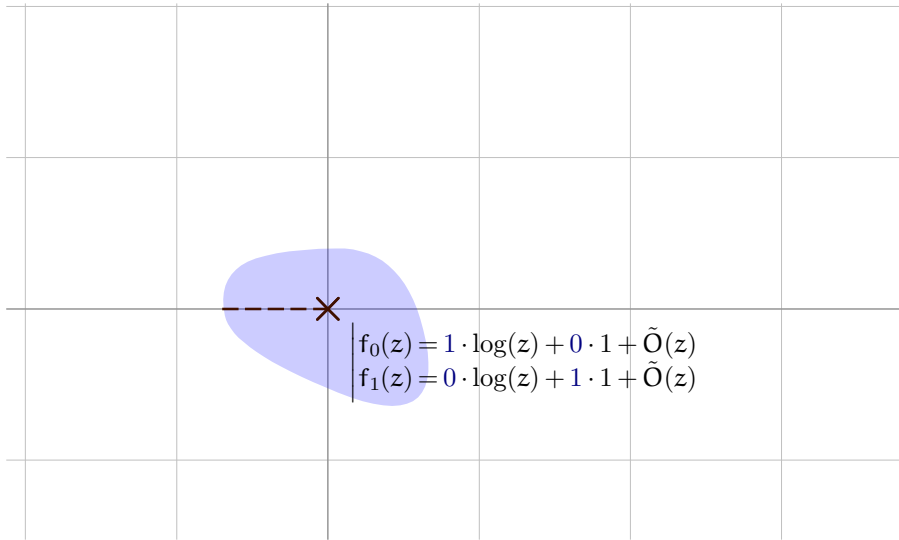
Initial Conditions at Regular Singular Points

$$\mathcal{L} = z Dz^2 + Dz + z$$



Initial Conditions at Regular Singular Points

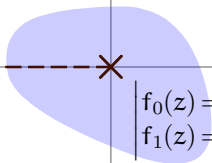
$$\mathcal{L} = z Dz^2 + Dz + z$$



$$\begin{cases} f_0(z) = 1 \cdot \log(z) + 0 \cdot 1 + \tilde{\mathcal{O}}(z) \\ f_1(z) = 0 \cdot \log(z) + 1 \cdot 1 + \tilde{\mathcal{O}}(z) \end{cases}$$

Initial Conditions at Regular Singular Points

$$\mathcal{L} = z Dz^2 + Dz + z$$



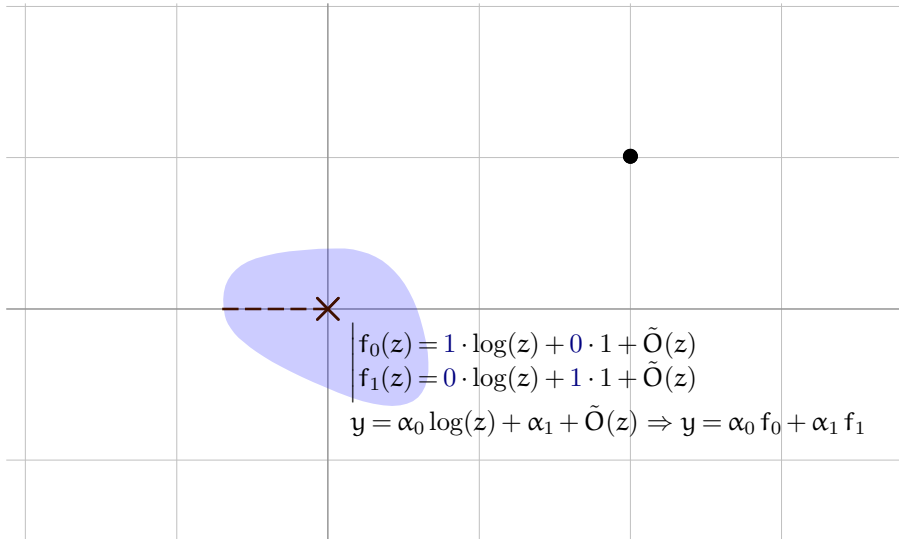
$$f_0(z) = 1 \cdot \log(z) + 0 \cdot 1 + \tilde{O}(z)$$

$$f_1(z) = 0 \cdot \log(z) + 1 \cdot 1 + \tilde{O}(z)$$

$$y = \alpha_0 \log(z) + \alpha_1 + \tilde{O}(z) \Rightarrow y = \alpha_0 f_0 + \alpha_1 f_1$$

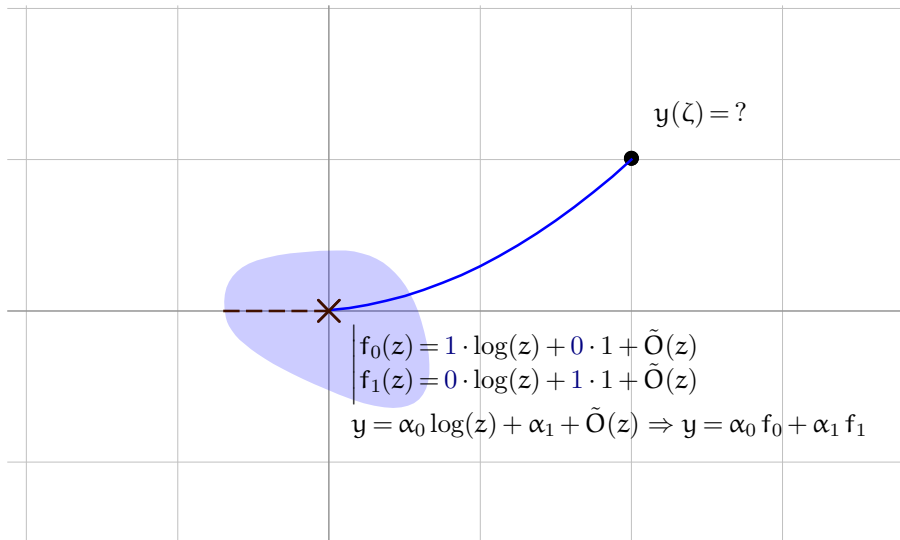
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Initial Conditions at Regular Singular Points

$$\mathcal{L} = z Dz^2 + Dz + z$$



Face-Centered Cubic Lattices ($d = 4$)

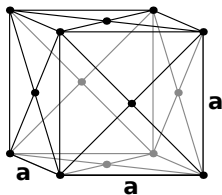
[Koutschan 2013]

$$P(z) = \frac{1}{\pi^4} \int \dots \int_{[0, \pi]^4} \frac{dk_1 dk_2 dk_3 dk_4}{1 - \frac{1}{6}(\cos k_1 \cos k_2 + \cos k_1 \cos k_3 + \dots + \cos k_3 \cos k_4)} z$$

$$\begin{aligned} \mathcal{L} = & (-1 + z) z^3 (2+z) (3+z) (6+z) (8+z) (4+3z)^2 Dz^4 \\ & + 2z^2 (4+3z) (-3456 - 2304z + 3676z^2 + 4920z^3 \\ & \quad + 2079z^4 + 356z^5 + 21z^6) Dz^3 \\ & + 6z (-5376 - 5248z + 11080z^2 + 25286z^3 \\ & \quad + 19898z^4 + 7432z^5 + 1286z^6 + 81z^7) Dz^2 \\ & + 12 (-384 + 224z + 3716z^2 + 7633z^3 + 6734z^4 \\ & \quad + 2939z^5 + 604z^6 + 45z^7) Dz \\ & + 12z (256 + 632z + 702z^2 + 382z^3 + 98z^4 + 9z^5) \end{aligned}$$


























$$\mathcal{L} \cdot P = 0$$

$$\text{r.p.} = 1 - \frac{1}{P(1)}$$



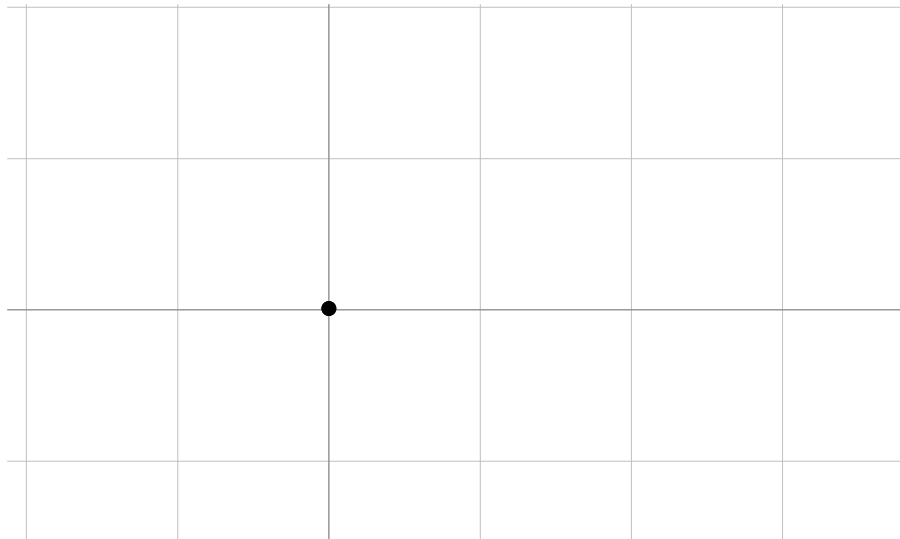
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[Bostan & Kauers '09; Melczer & Wilson '16; Bostan, Chyzak, van Hoeij, Kauers & Pech '17]

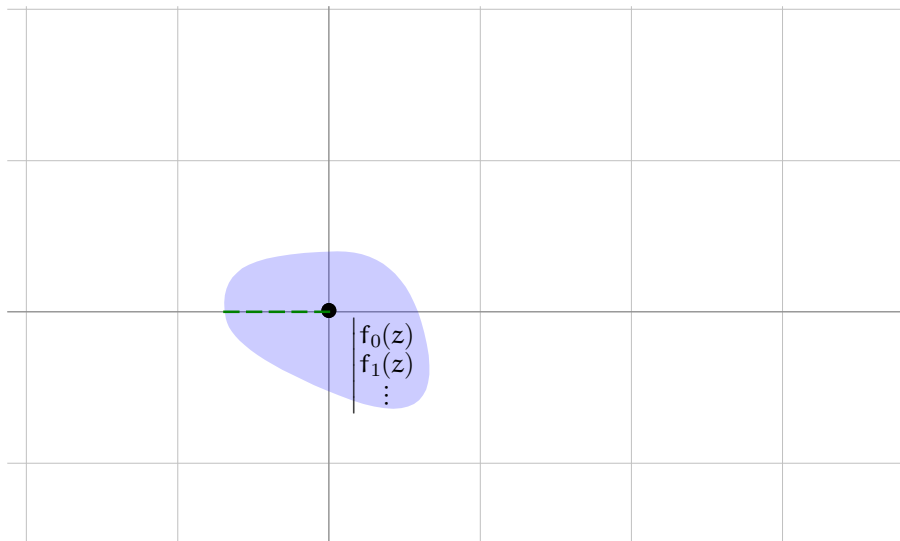
	OEIS		alg	asympt		OEIS		alg	asympt
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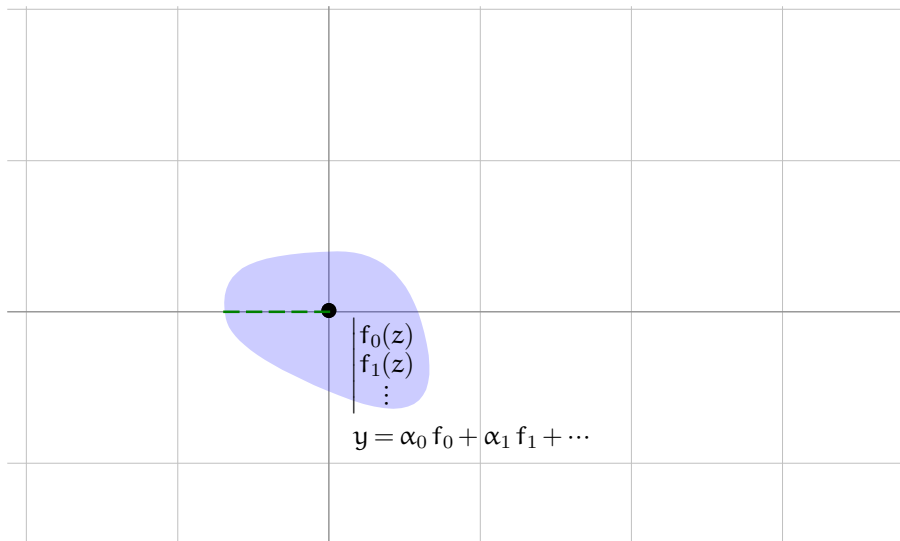
Transition Matrices



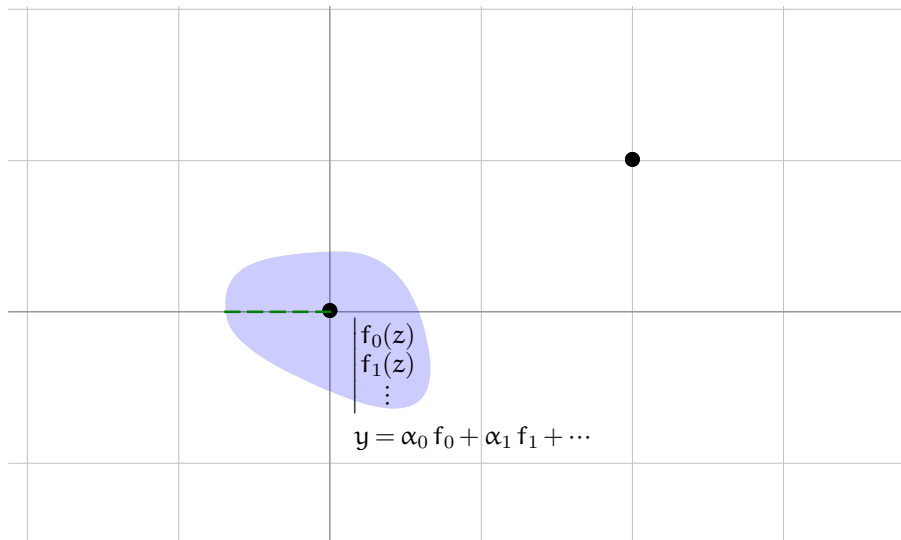
Transition Matrices



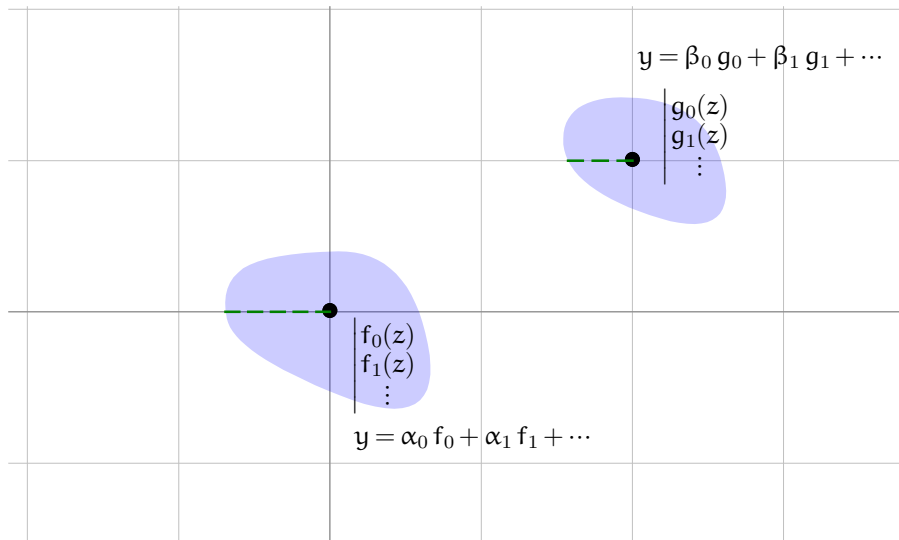
Transition Matrices



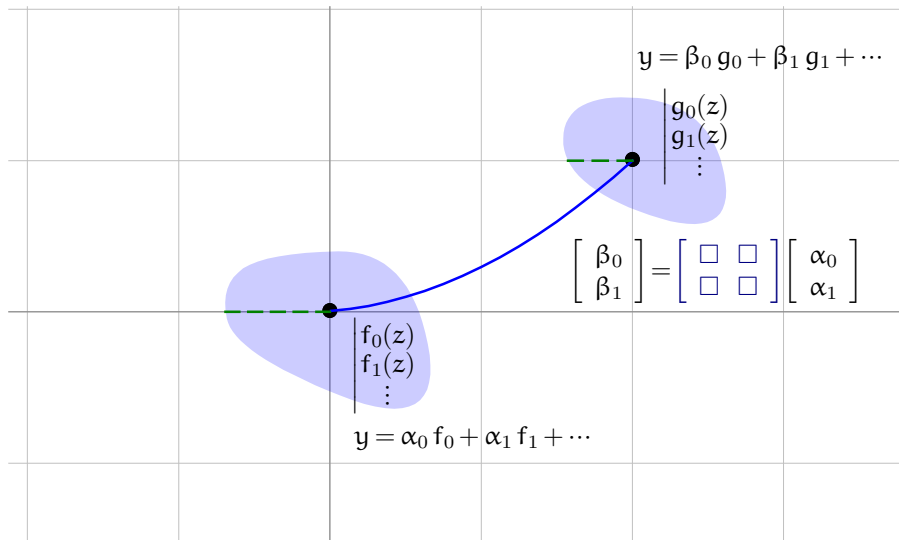
Transition Matrices



Transition Matrices



Transition Matrices





Summary

What it is: an extension of ore_algebra written in/for SageMath

What it does: numerical analytic continuation & singular connection, for arbitrary D-finite functions, with rigorous error bounds

How it works: Taylor series, analytic continuation, recurrences, adaptive majorants, ball arithmetic



Code available at

<http://kauers.de/software.html>

http://marc.mezzarobba.net/code/ore_algebra-analytic



Perspectives

Faster algorithms, lower-level code,

D-finite functions as objects,

irregular singular connection problems...

Bug reports, feature requests, examples welcome!

To Do: Features

Core features

- ✓ Analytic continuation
- ✓ Regular singular points
- Irregular singular points

Evaluation points

- ✓ Rationals, $\mathbb{Q}[i]$
- ✓ Algebraic numbers
- ✓ Small intervals
- Wide intervals
- Series ?

Initial values

- ✓ Numeric, interval
- ✓ Exact

User interface

- ✓ Evaluation
- ✓ Transition matrices
- Branch cuts
- Function objects

Applications

- ✓ Polynomial approximations
- Singularity analysis
- Zeros
- ...

To Do: Efficiency

▶ **Series summation algorithms**

- ▶ Floating-point summation with manual error analysis
- ▶ Rectangular splitting
- ▶ (Truncated) binary splitting @ reg. sing.
- ▶ Bit-burst method

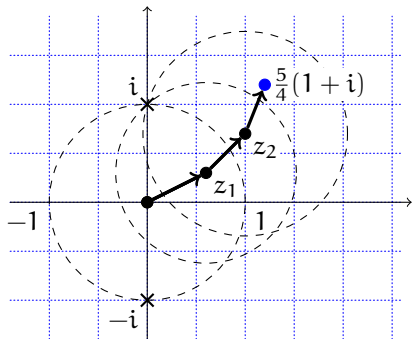
▶ **Analytic continuation algorithms**

- ▶ Automatic path optimization
- ▶ Analytic continuation of individual solutions
- ▶ Simultaneous computation of local solutions @ reg. sing.

▶ **Implementation**

- ▶ Compiled version of direct summation
- ▶ Optimizations
- ▶ Faster basic arithmetic (in Sage)

A Taylor Series Method



$$\arctan\left(\frac{5}{4}(1+i)\right) ?$$

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\dots + 0.22\dots \\ 0 & 0.72\dots - 0.20\dots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.39\dots + 0.24\dots \\ 0 & 0.57\dots - 0.29\dots \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$

...

- ▶ Locally, the solutions are given by **convergent power series**
- ▶ **Sum the series** numerically to get “initial values” at a new point
- ▶ Large steps (\propto radius of convergence)
- ▶ Extends to the regular singular case

Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$ obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n) y_{n+s} + \dots + b_1(n) y_{n+1} + b_0(n) y_n = 0.$$

(And conversely, for D-finite formal power series.)

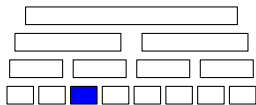
- ▶ Easy to generate
- ▶ Also leads to **fast algorithms**

[Schroepel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988;
van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

time $\mathbf{O}(\mathbf{M}(n \log^2 n))$, space $\mathbf{O}(n)$

for fixed z and $\varepsilon = 2^{-n}$



Recurrences

The **coefficients** of a D-finite function $\sum_{\nu \in \lambda + \mathbb{Z}} \sum_{k=0}^K y_{\nu,k} z^{\nu} \frac{\log(z)^k}{k!}$ obey a linear **recurrence relation** with polynomial coefficients:

$$[b_s(\nu + S_k) \cdot S_{\nu}^s + \dots + b_1(\nu + S_k) S_{\nu} + b_0(\nu + S_k)] \cdot (y_{\nu,k}) = 0.$$

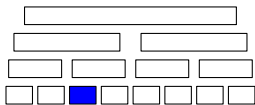
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Error Bounds

Round-off errors

Real & complex arithmetic based on **Arb**

[Johansson 2012-]

(`{Real, Complex}BallField` in Sage)

- ▶ More generally: takes care of error propagation
- ▶ Arb supports truncated power series (cf. autodiff)
- ▶ Manual error analysis still useful when intervals blow up

Truncation Errors

$$\sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\text{known}} + \underbrace{\sum_{n=N}^{\infty} u_n z^n}_{|\cdot| \leq ?}$$

- ▶ Majorant series
- ▶ "Adaptive" bounds using residuals

Asymptotics of Apéry Numbers

$$a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad b_n = \sum_{k=1}^n \left(\frac{a_n}{k^3} - \sum_{m=1}^k \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2 m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

(1, 5, 73, 1445, 33001...) (0, 6, 351/4, 62531/36, ...)

- ▶ The OGS $a(z)$ and $b(z)$ are solutions of $L = z^2(z^2 - 34z + 1)D_x^4 + \dots$
- ▶ Singular points: $0, \alpha = (\sqrt{2} + 1)^4 \approx 33.9, \alpha^{-1} = (\sqrt{2} - 1)^4 \approx 0.0294$
- ▶ Prove: $a_n, b_n = \alpha^{n+o(n)} \quad b_n - \zeta(3) a_n = \alpha^{-n+o(n)}$
- ▶ Local expansion at α^{-1} : $a(z) = c_0 f_0(z) + c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z)$
 where $f_0(\alpha^{-1} + t) = 1 + O(t^3) \quad f_3(\alpha^{-1} + t) = t + O(t^3)$
 $f_1(\alpha^{-1} + t) = \sqrt{t} + O(t^3) \quad f_4(\alpha^{-1} + t) = t^2 + O(t^3)$
- ▶ Singularity analysis: ($c_1 \neq 0$)

$$a(z) \sim c_1 \sqrt{z - \alpha^{-1}} \quad \Rightarrow \quad a_n \sim c_1 [z^n] \sqrt{z - \alpha^{-1}} \sim \frac{c_1 i}{2\sqrt{\alpha\pi}} \alpha^n n^{-3/2}$$

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