Hypergeometric Summation

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MPRI CompAlg, 2025–2026, Lecture 10

Introduction

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1$$

indefinite sum

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

definite sum

More examples:

$$\sum_{k=0}^{n} \binom{2k}{k} 4^{-k}, \quad \sum_{k=0}^{n} \frac{(4k+1)\,k!}{(2k+1)!}, \quad \sum_{k=0}^{n} \binom{n}{2k} \binom{2k}{k} 4^{-k}, \quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \ldots$$

Introduction

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1$$

indefinite sum

$$F(n+1) - F(n) = f(n)$$

indefinite integral

$$F' = f, \quad F(x) = \int_0^x f(t)\,dt$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

definite sum

$$F(n) = \sum_{k=k_0(n)}^{k_1(n)} f(n,k)$$

definite integral

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

More examples:

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1 Hypergeometric sequences, hypergeometric terms

Hypergeometric sequences

$\mathbb{K}$ — field of char. 0

Definition. A sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ is **hypergeometric** if there exists a rational function $a(x) \in \mathbb{K}(x)$ s.t.

$$u_{n+1} = a(n)\, u_n \quad \text{for all but finitely many } n \in \mathbb{N}.$$

In some references: for all n where $a(n)$ is defined.

Equivalently: if it is P-finite of order 1, i.e. if it satisfies a relation of the form

$$\forall n \in \mathbb{N}, \quad p(n)\, u_{n+1} = q(n)\, u_n, \qquad\qquad p, q \in \mathbb{K}[x].$$

As for general P-finite sequences, $p(x)$ and $q(x)$ may have common factors.

Examples. $n!$, $\quad 2^n$, $\quad 2^n n!$, $\quad \dfrac{n^2+1}{n+3}$, $\quad (42, 1, 57, 0, -100, 1, 2, 4, 8, 16, \ldots,)$

Non-example. $2^n + n!$

Closed form

$u_{n+1} = a(n)\, u_n$ for $n \geqslant n_0$

For $\mathbb{K} = \mathbb{C}$, write

$$a(x) = c\, \frac{(x-\xi_1)\cdots(x-\xi_\ell)}{(x-\zeta_1)\cdots(x-\zeta_m)}.$$

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Then, for $n \geqslant n_0$,

$$u_n \;=\; a(n-1)\, a(n-2) \cdots a(n_0)\, u_{n_0}$$

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Then, for $n \geqslant n_0$,

$$\begin{aligned} u_n &= a(n-1)\,a(n-2)\cdots a(n_0)\,u_{n_0} \\ &= u_{n_0}\,c^{n-n_0}\prod_{k=n_0}^{n-1}\frac{(k-\xi_1)\cdots(k-\xi_\ell)}{(k-\zeta_1)\cdots(k-\zeta_m)} \end{aligned}$$

Closed form

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Then, for $n \geqslant n_0$,

Recall:
$\Gamma(x+1) = x\,\Gamma(x)$
$\Gamma(n+1) = n!$

$$\begin{aligned}
u_n &= a(n-1)\,a(n-2)\cdots a(n_0)\,u_{n_0} \\
&= u_{n_0}\,c^{n-n_0}\prod_{k=n_0}^{n-1}\frac{(k-\xi_1)\cdots(k-\xi_\ell)}{(k-\zeta_1)\cdots(k-\zeta_m)} \\
&= u_{n_0}\,c^{n-n_0}\,\frac{\Gamma(n-\xi_1)\cdots\Gamma(n-\xi_\ell)}{\Gamma(n_0-\xi_1)\cdots\Gamma(n_0-\xi_\ell)}\,\frac{\Gamma(n_0-\zeta_1)\cdots\Gamma(n_0-\zeta_m)}{\Gamma(n-\zeta_1)\cdots\Gamma(n-\zeta_m)}
\end{aligned}$$

Closed form

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$$\begin{aligned} u_n &= a(n-1)\, a(n-2)\cdots a(n_0)\, u_{n_0} \\ &= u_{n_0}\, c^{n-n_0} \prod_{k=n_0}^{n-1} \frac{(k-\xi_1)\cdots(k-\xi_\ell)}{(k-\zeta_1)\cdots(k-\zeta_m)} \\ &= u_{n_0}\, c^{n-n_0} \frac{\Gamma(n-\xi_1)\cdots\Gamma(n-\xi_\ell)}{\Gamma(n_0-\xi_1)\cdots\Gamma(n_0-\xi_\ell)}\, \frac{\Gamma(n_0-\zeta_1)\cdots\Gamma(n_0-\zeta_m)}{\Gamma(n-\zeta_1)\cdots\Gamma(n-\zeta_m)} \\ &= A\, c^n \frac{\Gamma(n-\xi_1)\cdots\Gamma(n-\xi_\ell)}{\Gamma(n-\zeta_1)\cdots\Gamma(n-\zeta_m)} \end{aligned}$$

Conversely, a sequence of this form is hypergeometric.

Hypergeometric series

A **generalized hypergeometric series** is a power series whose coefficient sequence is hypergeometric. Notation:

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \, c\,x \right) = \sum_{n=0}^{\infty} u_n\, x^n $$

$$ \text{where} \qquad u_{n+1} = c\,\frac{\prod_i (n+a_i)}{(n+1)\prod_j (n+b_j)}\, u_n, \quad u_0 = 1. $$

Cornerstone of the classical theory of special functions:

- $(1-x)^a = {}_1F_0(-a;;x)$,
  $\ln(1+x) = x\,{}_2F_1(1,1;2;-x)$,
  $\mathrm{Li}_2(x) = x\,{}_3F_2(1,1,1;2,2;x)$,
  etc.

- Numerous identities, e.g.,
  ${}_2F_1(2\,a, 2\,b; a+b+\frac{1}{2}; x) = {}_2F_1(a, b; a+b+\frac{1}{2}; 4\,x\,(1-x))$ (Kummer)

Hypergeometric terms

"Definition". A **hypergeometric term** is a formal object $H(x)$ subject to the relation

$$H(x+1) = a(x)\,H(x) \qquad \text{for some } a(x) \in \mathbb{K}(x).$$

More formally:

- a **difference ring extension** of $\mathbb{K}(x)$ is a ring $R \supseteq \mathbb{K}(x)$
  with a ring endomorphism $\sigma\colon R \to R$
  s.t. $\sigma(f) = f(x+1)$ for $f \in \mathbb{K}(x)$
- a hypergeometric term is an element $H \in R$ s.t. $\sigma(H) = a\,H$ for some $a \in \mathbb{K}(x)$.

Example. In the ring $\mathbb{K}(x)[U]$, setting $\sigma(x) = x+1$ and $\sigma(U) = (x+1)\,U$ defines an ring endomorphism.

The element U is a hypergeometric term $\rightsquigarrow$ algebraic model of $(n!)_{n\in\mathbb{N}}$.

In this kind of situation we write u_n instead of U and u_{n+k} instead of $\sigma^k(U)$.

Similarity

Definition. Two hypergeometric terms u_n, v_n over $\mathbb{K}$ are **similar** if there exists $a \in \mathbb{K}(n)$ such that $u_n = a(n)\, v_n$.

Properties. Let u_n, v_n be hypergeometric terms.

- The sum $u_n + v_n$ is hypergeometric iff u_n and v_n are similar.
- For $L \in \mathbb{K}(n)[S_n]$ (recurrence operator), $L(u_n)$ is a hypergeometric term similar to u_n.

Example. With $u_n = n!$,

$$\begin{aligned} \frac{1}{n+7} u_{n+2} + n\, u_n &= \frac{1}{n+7} (n+2)\,(n+1)\, u_n + n\, u_n \\ &= (\text{rational})\, u_n \end{aligned}$$

Bivariate hypergeometric sequences and terms

“Definition”. A multivariate sequence/term is hypergeometric if it is hypergeometric w.r.t. each of the variables.

- A bivariate hypergeometric term is a formal object $H(x,y)$ subject to the relations

$$\begin{cases} H(x+1,y) = a(x,y)\,H(x,y) \\ H(x,y+1) = b(x,y)\,H(x,y) \end{cases} \qquad \text{for some } a,b \in \mathbb{K}(x,y).$$

- A bivariate sequence $(u_{n,k})_{n,k\in\mathbb{N}}$ is hypergeometric if $\exists a, b \in \mathbb{K}(x,y)$ s.t.

$$\begin{aligned} &\forall n \in \mathbb{N},\ \ \forall k \in \mathbb{N}\setminus\{\text{finite}\}, && u_{n+1,k} = a(n,k)\,u_{n,k} \\ &\forall k \in \mathbb{N},\ \ \forall n \in \mathbb{N}\setminus\{\text{finite}\}, && u_{n,k+1} = b(n,k)\,u_{n,k} \end{aligned}$$

Not “for all but finitely many (n,k)” — e.g., $a(n,k) = \dfrac{1}{n-k}$ is allowed

Example. $\displaystyle\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$

Remark. Compatibility condition: $b(n+1,k)\,a(n,k) = a(n,k+1)\,b(n,k)$

2 Indefinite summation

The indefinite summation problem

Let $(u_n)_{n\in\mathbb{N}}$ be hypergeometric over $\mathbb{K}$, with $u_{n+1} = a(n)\,u_n$.

The indefinite sum $s_n = \sum_{k=0}^{n-1} u_k$ is P-finite:

$$u_n = s_{n+1} - s_n \qquad \text{so} \qquad (s_{n+2} - s_{n+1}) - a(n)\,(s_{n+1} - s_n) = 0.$$

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Problem, sequence version. Given $a(x)$,

- Decide if the sequence (s_n) is hypergeometric,
- If so, compute $b(n)$ s.t. $s_{n+1} = b(n)\, s_n$. ($b(n)$ + initial value(s) $\rightsquigarrow$ closed form for s_n)

Problem, algebraic version. Given $a(n) \in \mathbb{K}(n)$:

- Decide if there exists a hypergeometric **term** s_n s.t. $s_{n+1} - s_n = u_n$,
- If so compute $b(n)$ s.t. $s_{n+1} = b(n)\, s_n$

Reduction to rational solutions

$u_{n+1} = a(n)\, u_n$
$s_{n+1} - s_n = u_n$

1. Suppose (s_n) is hypergeometric:

$$s_{n+1} = b(n)\, s_n$$

Then $s_{n+1} - s_n = u_n$ implies

$$b(n)\, s_n - s_n \;=\; u_n$$

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$$\begin{aligned} b(n)\, s_n - s_n &= u_n \\ s_n &= \lambda(n)\, u_n \end{aligned} \qquad \text{(similarity)}$$

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2. Rewrite the equation using the previous observation:

$$s_{n+1} - s_n = u_n$$

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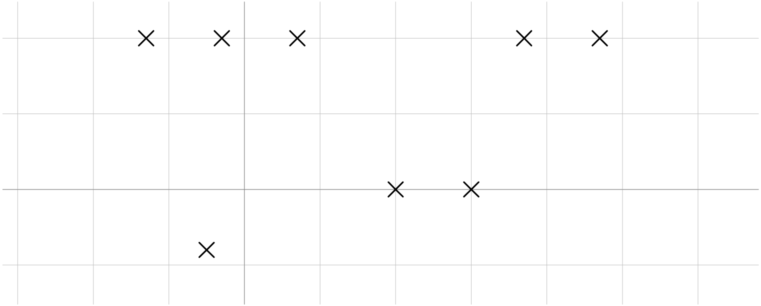
$$a(n)\, \lambda(n+1) - \lambda(n) = 1$$

"Gosper equation"
(inhom. recurrence)

A crude denominator bound



known $\downarrow$ unknown $\downarrow$

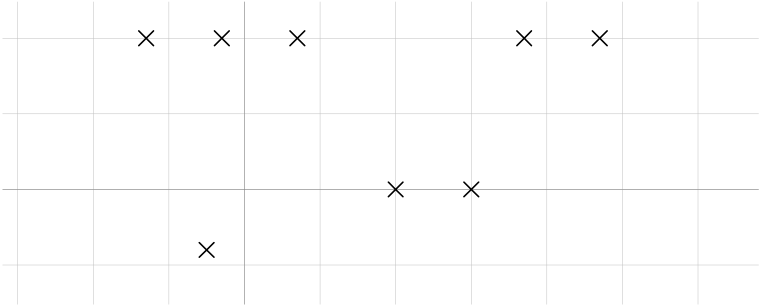
$$a(x)\,\lambda(x+1) - \lambda(x) = 1$$

$$a(x) = \frac{p(x)}{q(x)}$$

$(p \wedge q = 1)$

$\times$ poles of $\lambda(x)$

A crude denominator bound



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- Any pole of $\lambda(x)$ must be $\left|\begin{array}{l}\text{a zero of } q(x)\\ \text{or a pole of } \lambda(x+1)\end{array}\right.$

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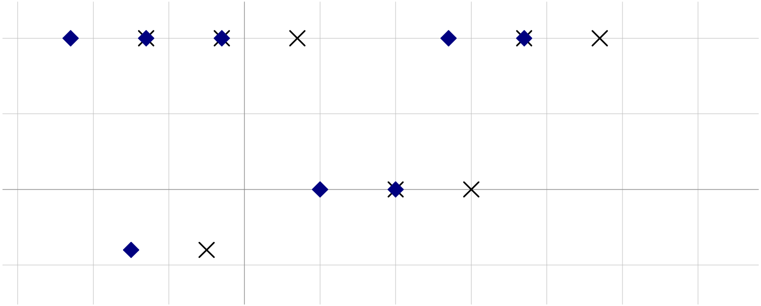

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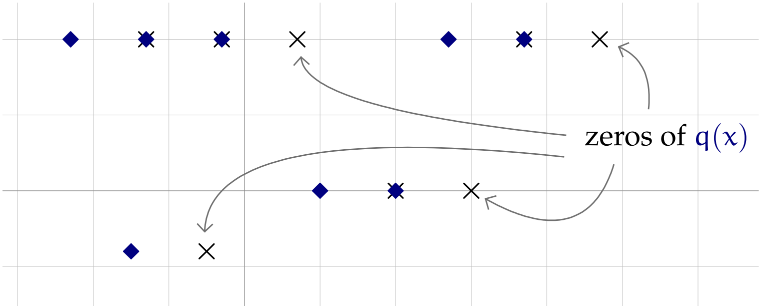

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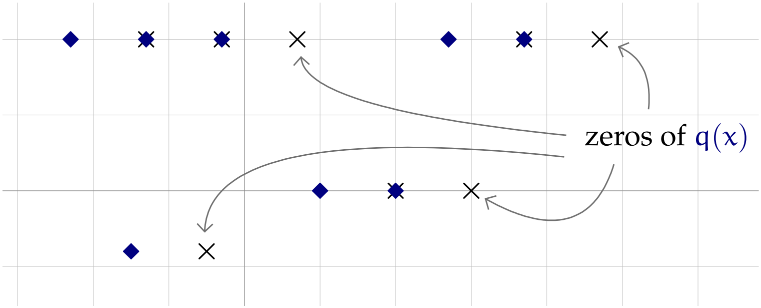

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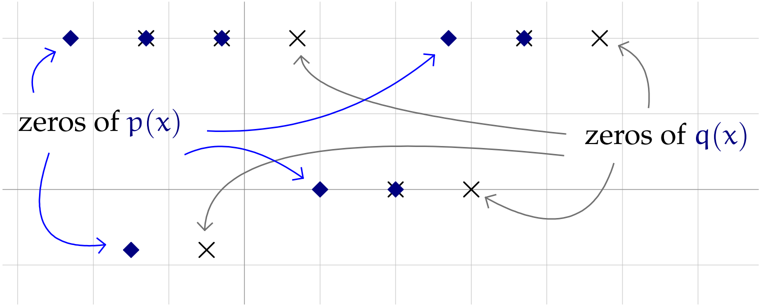

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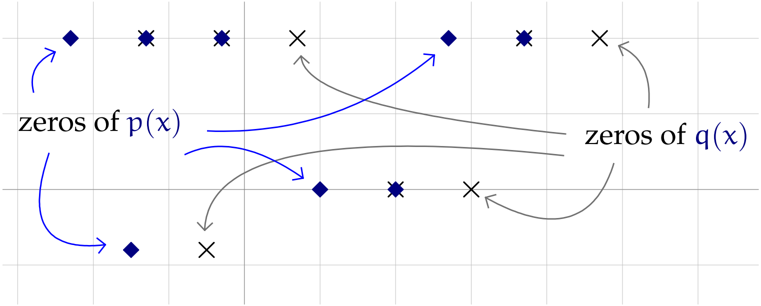

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➡ $\mathrm{den}(\lambda)$ divides $q(x)\,q(x+1)\cdots q(x+h-1)$ where $h = \max\{i : q(x+i) \wedge p(x) \neq 1\}$

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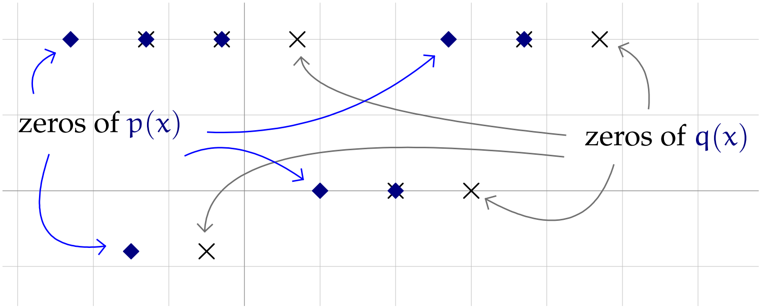

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➡ $\mathrm{den}(\lambda)$ divides $q(x)\,q(x+1)\cdots q(x+h-1)$ where $h = \max\{i : q(x+i) \wedge p(x) \neq 1\}$

h is a zero of $\mathrm{Res}_x(p(x), q(x+h))$

Gosper forms of rational functions

A pole of $\lambda(x)$ that is not a pole of $\lambda(x+1)$ is a zero of $q(x) \wedge p(x-i)$ for some $i \geqslant 1$.

If all these gcds are 1, any rational solution is polynomial.

We can reduce to this case!

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$$\begin{aligned} a(x) &= \frac{p(x)}{q(x)} \\ &= \frac{p_1(x)}{q_1(x)} \frac{f_1(x+i_1)}{f_1(x)} \end{aligned} \qquad f_1(x) = p(x-i_1) \wedge q(x) \neq 1$$

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&= \frac{p_1(x)}{q_1(x)} \frac{f_1(x+1) \cdots f_1(x+i_1-1)\, f_1(x+i_1)}{f_1(x)\, f_1(x+1) \cdots f_1(x+i_1-1)}
\end{aligned}$$

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&= \frac{p_1(x)}{q_1(x)} \frac{r_1(x+1)}{r_1(x)}
\end{aligned}$$

Gosper forms of rational functions

A pole of $\lambda(x)$ that is not a pole of $\lambda(x+1)$ is a zero of $q(x) \wedge p(x-i)$ for some $i \geqslant 1$.

If all these gcds are 1, any rational solution is polynomial.

We can reduce to this case!

$$\begin{aligned}
a(x) &= \frac{p(x)}{q(x)} && f_1(x) = p(x-i_1) \wedge q(x) \neq 1 \\
&= \frac{p_1(x)}{q_1(x)} \frac{f_1(x+i_1)}{f_1(x)} \\
&= \frac{p_1(x)}{q_1(x)} \frac{f_1(x+1) \cdots f_1(x+i_1-1)\, f_1(x+i_1)}{f_1(x)\, f_1(x+1) \cdots f_1(x+i_1-1)} \\
&= \frac{p_1(x)}{q_1(x)} \frac{r_1(x+1)}{r_1(x)} && f_2(x) = p_1(x-i_2) \wedge q_1(x) \neq 1 \\
&= \cdots
\end{aligned}$$

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More classical notation: $(Q(x)\,/\,R(x+1))\,(P(x+1)\,/\,P(x))$.

Improved reduction to polynomial solutions

$$a(x)\,\lambda(x+1)-\lambda(x)=1$$

$$a(x)=\frac{P(x)}{Q(x)}\,\frac{R(x+1)}{R(x)}$$
$$\forall i\geqslant 0,\quad P(x-i)\wedge Q(x)=1$$

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The equation becomes

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Letting $\mu(x)=R(x)\,\lambda(x)$:

$$\frac{P(x)}{Q(x)}\,\mu(x+1)-\mu(x)=R(x).$$

Then all rational solutions $\mu(x)$ are polynomial.

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Then all rational solutions $\mu(x)$ are polynomial.

Additionally $Q(x)$ must divide $\mu(x+1)$:

$$P(x)\,\varphi(x+1)-Q(x-1)\,\varphi(x)=R(x) \qquad \varphi(x)=\frac{\mu(x)}{Q(x-1)}\in\mathbb{K}[x].$$

Polynomial solutions

Polynomial solutions

$$f(x)\,\varphi(x+1) - g(x)\,\varphi(x) = h(x)$$

$$f, g, h, \varphi \in \mathbb{K}[x]$$

- Easy case: if $\deg f \neq \deg g$ or $\operatorname{lt}(f) \neq \operatorname{lt}(g)$, then

$$\deg \varphi = \deg h - \max(\deg f, \deg g)$$

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$$f_d = g_d$$

$$\begin{aligned} f(x) &= f_d x^d + \cdots \\ g(x) &= g_d x^d + \cdots \\ \varphi(x) &= \varphi_\delta x^\delta + \cdots \end{aligned}$$

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$$\underbrace{f(x)\,\Delta\varphi(x)} \quad + \quad \underbrace{(f(x) - g(x))\,\varphi(x)} \quad = h(x)$$

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$$\underbrace{f(x)\,\Delta\varphi(x)}_{f_d \delta \varphi_\delta\, x^{d+\delta-1} + \cdots} \quad + \quad \underbrace{(f(x) - g(x))\,\varphi(x)}_{(f_{d-1} - g_{d-1})\,\varphi_\delta\, x^{d-1+\delta} + \cdots} \quad = h(x)$$

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If $d + \delta - 1 > \deg h$ then $\delta = (g_{d-1} - f_{d-1})/f_d$.

Summary

Algorithm. *Input*: a hypgeom. term u_n *Output*: s_n s.t. $s_{n+1} - s_n = u_n$ or None

1. Write $a(n) = u_{n+1}/u_n$ in Gosper form $\dfrac{P(n)}{Q(n)} \dfrac{R(n+1)}{R(n)}$.

2. Solve $P(x)\,\varphi(x+1) - Q(x-1)\,\varphi(x) = R(x)$ for $\varphi \in \mathbb{K}[x]$.

3. If there is a solution $\varphi(x)$, return $\dfrac{R(n)\,\varphi(n)}{Q(n-1)} u_n$.

   Otherwise, return None.

[Gosper 1978]

Theorem. If the algorithm returns None, then the hypergeometric term u_n has no hypergeometric indefinite sum.

Remark. Similar ideas ➡ algos for | polynomial solutions of recurrences of order >1
rational

Back to sequences

Let $(u_n)_{n\in\mathbb{N}}$ be a hypergeometric **sequence** with $u_{n+1} = a(n)\, u_n$ for $n \in \mathbb{N} \setminus \Sigma$.

$$\sum_{k=0}^{n-1} u_k \ =$$

Back to sequences

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$\Sigma' = \{\text{poles of } a(x), \lambda(x), \lambda(x+1)\}$

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$\Sigma' = \{\text{poles of } a(x), \lambda(x), \lambda(x+1)\}$

For $n \in \mathbb{N}\setminus(\Sigma\cup\Sigma')$, $\quad u_n \;=\; \lambda(n+1)\,u_{n+1} - \lambda(n)\,u_n \quad$ (mult. by u_n)

$$\sum_{k=0}^{n-1} u_k \;=$$

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$$\begin{aligned} u_n &= \lambda(n+1)\, u_{n+1} - \lambda(n)\, u_n && \text{(mult. by } u_n\text{)} \\ &= s_{n+1} - s_n && \text{where } s_n := \lambda(n)\, u_n \end{aligned}$$

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$$\sum_{k=0}^{n-1} u_k \;=\; \sum_{k=0}^{\sigma_1-1} u_k + u_{\sigma_1} + \sum_{k=\sigma_1+1}^{\sigma_2-1} u_k + u_{\sigma_2} + \cdots + \sum_{k=\sigma_m+1}^{n} u_k$$

where $\Sigma\cup\Sigma' = \{\sigma_1 < \sigma_2 < \cdots\}$
$\sigma_m < n \leqslant \sigma_{m+1}$

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If the algorithm finds nothing, $\nexists$ hypergeom. $(s_n)_{n\in\mathbb{N}}$ with $s_n = c + \sum_{k=0}^{n-1} u_k$ infinitely often, except maybe if $s_n = 0$ for large n. (?)

3 Intermezzo: parametrized definite summation

Parametrized definite summation

Problem. Given m similar hypergeometric terms $r_1(n)\,u_n, \ldots, r_m(n)\,u_n$,

find | constants $\alpha_1, \ldots, \alpha_m$ such that
| a hypergeometric term s_n

$$s_{n+1} - s_n = (\alpha_1\, r_1(n) + \cdots + \alpha_m\, r_m(n))\, u_n$$

(or prove that no solution exists).

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(or prove that no solution exists).

Handwaving. Gosper's algorithm works with minor adaptations.

Gosper's equation becomes

$$a(x)\,\lambda(x+1) - \lambda(x) = \alpha_1\, r_1(x) + \cdots + \alpha_m\, r_m(x)$$

$u_{n+1} = a(n)\,u_n$
$s_n = \lambda(n)\,u_n$

Denominator valid for any $(\alpha_1,\dots,\alpha_m)$ by taking into account the poles of $r_1(x), \dots, r_m(x)$; similarly for degree bounds.

Searching for polynomial solutions leads to equations linear in the coefficients of the polynomial **and in the α_i**; solve for both simultaneously.

4 Definite summation

Definite hypergeometric summation

General goal: “compute” $\displaystyle\sum_{k=k_0(n)}^{k_1(n)} u_{n,k}$ where $u_{n,k}$ is a hypergeometric.

Here:
- ”compute” = find a recurrence
- natural boundaries: $\displaystyle\sum_k u_{n,k}$ over “all possible k”

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=-\infty}^{\infty} \binom{n}{k}$$

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Problem. Given a bivariate hypergeometric sequence $u_{n,k}$

$$F(n) = \sum_k u_{n,k},$$

find (if possible) a linear recurrence

$$a_s(n)\, F(n+s) + \cdots + a_1(n)\, F(n+1) + a_0(n)\, F(n) = 0, \qquad a_i \in \mathbb{K}(x).$$

(E.g., a recurrence $a_s(n)\, u_{n+s,k} + \cdots + a_0(n)\, u_{n,k} = 0$ with coeff independent of k for the summand would work for the sum…)

Creative Telescoping

Key idea. Suppose that we can find operators $P(n, S_n)$ and $Q(n, S_n, k, S_k)$ s.t.

$$\underbrace{P(n, S_n)}_{\text{telescoper}} \cdot u_{n,k} = (S_k - 1) \underbrace{Q(n, S_n, k, S_k)}_{\text{certificate}} \cdot u_{n,k}.$$

Creative Telescoping

Operator notation: $S_n \cdot u_{n,k} = u_{n+1,k}, \quad S_k \cdot u_{n,k} = u_{n,k+1}$

E.g., $u_{n,k}$ is a biv. hypgeom term iff $\exists a, b \in \mathbb{K}(n,k)$ s.t. $(S_n - a(n,k)) \cdot u_{n,k} = 0$
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$$P(n, S_n) \cdot \sum_k u_{n,k} = \sum_k \left(v_{n,k+1} - v_{n,k} \right) = 0.$$

$$v_{n,k} = Q(n, S_n, k, S_k) \cdot u_{n,k}$$

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$v_{n,k} = Q(n, S_n, k, S_k) \cdot u_{n,k}$

Algebraic problem: given a bivariate hypergeometric **term** $u_{n,k}$, find P, Q as above.

For **sequences**, check poles, natural boundaries, adapt if necessary

Example

$$u_{n,k} = \binom{n}{k}$$

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&= \left(\frac{k+1}{n-k}\,\frac{n-k}{k+1}-\frac{k}{n+1-k}\right)u_{n,k}\\
[(S_n-2)-(S_k-1)\,(S_n-1)]\cdot u_{n,k} &= 0\\
(S_n-2)\cdot\sum_n u_{n,k} &= 0
\end{aligned}$$

Zeilberger's algorithm: idea

$$\underbrace{P(n, S_n)}_{\text{telescoper}} \cdot u_{n,k} = (S_k - 1) \underbrace{Q(n, S_n, k, S_k)}_{\text{certificate}} \cdot u_{n,k}$$

Choose an order s and make an ansatz

$$P(n, S_n) = a_s(n)\, S_n^s + \cdots + a_1(n)\, S_n + a_0(n) \, .$$

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The creative telescoping equation becomes

$$(S_k - 1) \cdot \underset{\uparrow}{v_{n,k}} = \overset{\downarrow}{a_s(n)}\ \underbrace{b_s(n,k)\, u_{n,k}} + \cdots + \overset{\downarrow}{a_1(n)}\ \underbrace{b_1(n,k)\, u_{n,k}} + \overset{\downarrow}{a_0(n)}\ \underbrace{u_{n,k}}$$

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unknown constants (w.r.t. k)

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This is a **parametrized definite summation** problem with $\mathbb{K}(n)$ as a base field.

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Choose an order s and make an ansatz $\underbrace{\boxed{P(n, S_n)}}_{\text{telescoper}} \cdot u_{n,k} = (S_k - 1) \underbrace{\boxed{Q(n, S_n, k, S_k)}}_{\text{certificate}} \cdot u_{n,k}$

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This is a **parametrized definite summation** problem with $\mathbb{K}(n)$ as a base field.

If there is a solution $(a_1, \ldots, a_s)$ s.t. $v_{n,k} = \lambda(n,k)\, u_{n,k}$, the parametrized Gosper algorithm can find it.

Zeilberger's algorithm [Zeilberger, 1990]

Algorithm. *Input*: a biv. hypgeom. term $u_{n,k}$ *Output*: operators $P(n, S_n)$ and $Q(n,k)$

For $i = 0, 1, 2, \ldots$:

1. Call the parametrized Gosper algorithm over $\mathbb{K}(n)$ to search for $a_0, \ldots, a_s \in \mathbb{K}(n)$ and $v_{n,k} = \lambda(n,k)\, u_{n,k}$ hypergeometric w.r.t. k over $\mathbb{K}(n)$ such that

$$v_{n,k+1} - v_{n,k} = a_s(n)\, u_{n+s,k} + \cdots + a_1(n)\, u_{n+1,k} + a_0(n)\, u_{n,k}$$

2. If a solution was found, return

$$P(n, S_n) = a_s(n)\, S_n^s + \cdots + a_1(n)\, S_n + a_0(n), \quad Q(n,k) = \lambda(n,k)$$

Theorem. The output (if any) satisfies $P(n, S_n) = [(S_k - 1)\, Q(n,k)] \cdot u_{n,k}$

Termination

Definition. A **proper hypergeometric term** is a hypergeometric term of the form

$$u_{n,k} = \underbrace{p(n,k)}_{\in\mathbb{K}(n,k)} \underset{\in\mathbb{K}}{\underset{\uparrow}{Z}}{}^{k} \bigl(\underset{\in\mathbb{Z}}{\underset{\uparrow}{a_1}}\, n + \underset{\in\mathbb{Z}}{\underset{\uparrow}{b_1}}\, k + \underset{\in\mathbb{Z}}{\underset{\uparrow}{c_1}}\bigr)!^{\overset{\in\mathbb{Z}}{\overset{\downarrow}{e_1}}} \cdots \bigl(\underset{\in\mathbb{Z}}{\underset{\uparrow}{a_\ell}}\, n + \underset{\in\mathbb{Z}}{\underset{\uparrow}{b_\ell}}\, k + \underset{\in\mathbb{Z}}{\underset{\uparrow}{c_\ell}}\bigr)!^{\overset{\in\mathbb{Z}}{\overset{\downarrow}{e_\ell}}}$$

Theorem. [Wilf, Zeilberger] Zeilberger's algorithm terminates when the input is a proper hypergeometric term.

This is only a sufficient condition.

Summary

$$
\begin{array}{ll}
\text{indefinite summation} & \\
\quad\downarrow & \text{creative telescoping} \\
\text{(parametrized) definite summation} & \\
\quad\downarrow & \text{Gosper equation} \\
\text{(parametrized) rational solutions} & \\
\quad\downarrow & \text{poles, Gosper form} \\
\text{(parametrized) polynomial solutions} & \\
\quad\downarrow & \text{finite differences} \\
\text{linear algebra} &
\end{array}
$$

Beyond hypergeometric sequences

- Differential (and other) analogues

$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{t^2} - t^2\right) dt$$

$$\left(P(x, D_x) - D_t\, Q(x, D_x, t, D_t) \right) \cdot f(x,t) = 0$$

- D-finite functions in several variables

  Including mixed diff./shift/…

$$\begin{cases} x\, J_n'(x) + x\, J_{n+1}(x) - n\, J_n(x) = 0 \\ x\, J_{n+2}(x) - 2\,(n+1)\, J_{n+1}(x) + x\, J_n(x) = 0 \end{cases}$$

$\rightarrow$ Gröbner bases of ideals of operators

- The idea of creative telescoping generalizes

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k} = \left(\frac{n}{2}+1\right)2^{3n} - 3n\,2^{n-2}\binom{2n}{n}$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx)\, dx = \frac{1}{2p}\exp\left(\frac{c^2-b^2}{4p}\right) J_n\left(\frac{bc}{p}\right)$$

$\rightarrow$ new closure property for D-finite functions! (under assumptions)

- Main difficulty: efficient algorithms for finding telescopers