Polynomial and Rational Solutions of Differential Equations

Problem. Given a linear ordinary differential equation

$$L(y)(x) = a_r(x)\, y^{(r)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0 \qquad \text{(DEQ)}$$

with $\left| \begin{array}{l} a_i \in \mathbb{K}[x] \text{ over a "suitable" field } \mathbb{K} \text{ of char } 0 \\ \deg a_i \leqslant d, \end{array} \right.$

compute a basis of the

→ polynomial solutions $y(x) \in \mathbb{K}[x]$

→ rational solutions $y(x) \in \mathbb{K}(x)$

→ (formal Laurent series solutions $y(x) \in \mathbb{K}((x))$). ◃

(These are indeed vector spaces!)

Motivation:

- useful on their own
- building block for other kinds of closed-form solutions and related problems

Example. Proving that $\exp(x^2)$ has no elementary antiderivative reduces to checking that $y'(x) + 2\,x\,y(x) = 1$ has no rational solution. ◃

1 Reminders/complements on series solutions

Recall from the previous lecture:

Proposition. The formal bilateral series $y(x) = \sum_{n=-\infty}^{\infty} y_n\, x^n$ satisfies $L(y) = 0$ iff

$$\forall n \in \mathbb{Z}, \qquad b_s(n)\, y_{n+s} + \cdots + b_0(n)\, y_n = 0 \qquad \text{(REC)}$$

where the b_i are obtained by changing $x \mapsto S^{-1}$, $\frac{\mathrm{d}}{\mathrm{d}x} \mapsto S \cdot N$ in (DEQ), that is,

$$b_n(N)\, S^s + \cdots + b_1(N)\, S + b_0(N) = S^{\delta}\,(a_r(S^{-1})\,(S N)^r + \cdots + a_1(S^{-1})\, S N + a_0(S^{-1}))$$

where $S N = (N+1)\, S$ and for some integer δ. ◃

Remark. It follows that $\deg(b_i) \leqslant r$. ◃

Definition. A point x_0 is called an *ordinary point* of L if $a_r(x_0) \neq 0$, a *singular point* if $a_r(0) = 0$. ◃

Proposition. If 0 is an ordinary point of L, then for any $(c_0, \ldots, c_{r-1}) \in \mathbb{K}^{r-1}$, there exists a solution $y \in \mathbb{K}[[x]]$ of $L(y) = 0$ such that

$$y(0) = c_0, \quad y'(0) = c_1, \quad \ldots, \quad y^{(r-1)}(0) = c_{r-1}. \qquad ◃$$

In other words, there are r linearly independent series solutions

$$\begin{array}{llll} 1 & & + \square x^r & + \cdots \\ & x & + \square x^r & + \cdots \\ & \ddots & & \\ & & x^{r-1} + \square x^r & + \cdots . \end{array}$$

Proof. The equation $L(y)=0$ is equivalent to

$$\begin{bmatrix} y'(x) \\ \vdots \\ y^{(r-1)}(x) \\ y^{(r)}(x) \end{bmatrix} = \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\frac{a_0(x)}{a_r(x)} & \cdots & \cdots & -\frac{a_{r-1}(x)}{a_r(x)} \end{bmatrix}}_{A(x)} \underbrace{\begin{bmatrix} y(x) \\ \vdots \\ y^{(r-2)}(x) \\ y^{(r-1)}(x) \end{bmatrix}}_{Y(x)}.$$

Make an ansatz for a solution $Y(x)=\sum_{n=0}^{\infty} Y_n x^n$. Since $a_r(0)\neq 0$, one can expand $A(x)$ as $\sum_{n=0}^{\infty} A_n x^n$. Then the equation rewrites as

$$\sum_{n=0} (n+1)\, Y_{n+1}\, x^n = \sum_{i=0}^{\infty} A_i x^i \sum_{j=0}^{\infty} Y_j x^j$$

which is equivalent to

$$\forall n \in \mathbb{N}, \quad (n+1)\, Y_{n+1} = \sum_{i=0}^{n} A_i\, Y_{n-i}.$$

So any $Y_0 = \begin{bmatrix} c_0 \\ \vdots \\ c_{r-1} \end{bmatrix}$ gives rise to a solution.

□

2 Polynomial solutions

Suppose that we had a bound D on the possible degrees of polynomial solutions. (Such a bound exists if $L\neq 0$.)

Algorithm. (Polynomial solutions given degree bound, slow)

- Make an ansatz

$$y(x) = y_0 + \cdots + y_D x^D.$$

- Plug it into the equation, leading to

$$\begin{aligned} L(y)(x) &= \sum_{i=0}^{r} \underbrace{\sum_{j=0}^{d} a_{i,j} x^j}_{a_i(x)} \sum_{k=0}^{D} y_k x^k \\ &= \sum_{n=0}^{D+d} [\text{linear expression in } y_0, \ldots, y_D]_n\, x^n. \end{aligned}$$

- Identify coefficients and solve the resulting linear system ($D+d+1$ equations, $D+1$ unknowns, overdetermined). ◃

It remains to see how to compute a suitable D.

For this, observe that a polynomial solution of $L(y)=0$ is the same as a *solution with finite support*

$$(y_0, y_1, \dots, y_N, 0, 0, 0, \dots) \qquad (*)$$

of the corresponding recurrence.

Lemma. For a solution of (REC) of the form ($*$) with $y_N \neq 0$ to exist, the index N must be a root of the polynomial b_0. ◃

Proof. Otherwise

$$y_N = -\frac{1}{b_0(N)} (b_1(N) \underbrace{y_{N+1}}_{0} + \cdots + b_s(N) \underbrace{y_{N+s}}_{0}) = 0. \qquad \square$$

So the roots of b_0 give the possible degrees of polynomial solutions of (DEQ), and we can compute a basis of these solutions with the following algorithm.

Algorithm. (Polynomial solutions, slow)

- Compute the coefficient b_0 of the recurrence associated with L.
- Compute the largest $D \in \mathbb{N}$ s.t. $b_0(D) = 0$ (if any, otherwise return $\{0\}$).
- Continue as above. ◃

Remark. This constrains the field $\mathbb{K}$: indeed, there are effective fields with no algorithm for computing the integer roots of polynomials. ◃

Remark.

- Polynomial solutions can be large. For instance, the polynomial

  $$y(x) = (1+x)^{1000}$$

  satisfies the small equation

  $$\frac{y'(x)}{y(x)} = \frac{1000}{1+x} \qquad \text{i.e.} \qquad (1+x)\, y'(x) - 1000\, y(x) = 0.$$

- The bound D can be large even in the absence of polynomial solutions. For instance

  $$y(x) = x^{1000}\, e^{1/x} \qquad \text{satisfies} \qquad \frac{y'(x)}{y(x)} = \frac{1000}{x} - \frac{1}{x^2}.$$

  This equation has order 1 so it cannot have a polynomial solution in addition to $y(x)$, but one can check (exercise!) that it leads to $D = 1000$ too. ◃

We will not analyze the complexity of this algorithm with respect to the size of the input, due in part to the dependency on polynomial roots. However, it is annoying that one has to solve a system of size $\approx D$. Can we do better?

3 Laurent series solutions

Proposition. Suppose that $L(y)=0$ has a solution of the form

$$\sum_{n\geqslant\nu} y_n x^n \qquad \text{with} \qquad \nu\in\mathbb{Z}, \quad y_\nu \neq 0.$$

Then ν is a root of $b_s(n-s)$. ◃

Proof. Otherwise, evaluating (REC) at $n=\nu-s$, we get

$$y_\nu = -\frac{1}{b_s(\nu-s)}(b_{s-1}(\nu-s)\, y_{\nu-1} + \cdots + b_0(\nu-s)\, y_{\nu-s}) = 0. \qquad \square$$

Definition. The polynomial $q_0(n)=b_s(n-s)$ is called the *indicial poynomial at* 0 of L. The corresponding polynomial after the change of variable $x=\xi+z$ in L is called the indicial polynomial at ξ. The polynomial $b_0(n)$ is called the *indicial polynomial at infinity*. ◃

One can check that the indicial polynomials at zero and infinity switch roles after the change of variable $z=1/x$ (hence the phrase "at infinity").

Algorithm. (Laurent series solutions, slow)

- Compute $\lambda=$ smallest root of q_0 in $\mathbb{Z}$
  $\mu=$ largest root of q_0 in $\mathbb{Z}$.
- Put $y(x)=y_\lambda x^\lambda+\cdots+y_\mu x^\mu$, substitute into $L(y)=\mathrm{O}(x^{\mu+d-r+1})$.
  [Here "$+\mathrm{O}(x^{\mu+d-r+1})$" is not the same as "mod $x^{\mu+d-r+1}$", because the ideal of $\mathbb{K}((x))$ generated by $x^{\mu+d-r+1}$ is $\mathbb{K}((x))$ itself…]
- Solve the resulting linear system. ◃

Proposition. This computes a basis of the space of Laurent series solutions, with each basis element truncated. ◃

Proof sketch. Any Laurent series solution satisfies the equation because any Laurent series solution has valuation $\geqslant\lambda$ and the coefficients of degree $\leqslant\mu+d-r$ of $L(y)$ only depend on $y_\lambda,\ldots,y_\mu$ (not on $y_{\mu+1},y_{\mu+2},\ldots$).
Since q_0 has no integer roots $>\mu$, given any approximate solution $y_\lambda x^\lambda+\cdots+y_\mu x^\mu+\mathrm{O}(x^{\mu+1})$, one can define $y_{\mu+1},y_{\mu+2},\ldots$ using (REC) and get a power series that satisfies the differential equation. So any approximate solution found by this algorithm is the initial part of a genuine solution. $\square$

At ordinary points, the situation is much simpler, and any solution can be computed from $(y_0,\ldots,y_{r-1})$ alone without hitting any singular index of the recurrence, as shown by the following proposition.

Proposition. At an ordinary point, the indicial polynomial is of the form

$$\text{constant}\cdot n\,(n-1)\cdots(n-r+1). \qquad \lhd$$

Proof. Since there are solutions of the form $x^k + \mathrm{O}(x^r)$ for $k=0,\ldots,r-1$, all these k must be roots, but we have also seen that the coefficient of (REC) have degree $\leqslant r$. □

4 Faster algorithms

More generally, when passing a singular index n:

$$\begin{aligned}
\underbrace{b_s(n-1)}_{\neq 0}\,y_{n+s-1} &= \ldots\\
\underbrace{b_s(n)}_{=0}\,y_{n+s} &= -[b_{s-1}(n)\,y_{n+s-1}+\cdots+b_0(n)\,y_n]\\
\underbrace{b_s(n+1)}_{\neq 0}\,y_{n+s+1} &= \ldots
\end{aligned}$$

two things happen:

→ free choice of y_{n+s}

→ **but** constraint on $y_n,\ldots,y_{n+s-1}$ (and thus on the sequence as a whole).

As long as we take this into account we can use the recurrence even in the presence of singular indices.

Exercise. Find the dimension of the space of solutions in $\mathbb{Q}^{\mathbb{Z}}$ of

$$(n-1)\,(n-2)\,u_n = u_{n-1} + (n-2)\,u_{n-2}$$

that are ultimately zero as $n \to -\infty$. ⊲

Solution. The first nonzero term of such a solution must be u_1 or u_2.

$$\begin{array}{llcllcl}
 & & & \vdots & & & \\
(n=0) & 2\,u_0 & = & u_{-1} & - & 2\,u_{-2}\\
(n=1) & 0 & = & u_0 & - & u_{-1}\\
(n=2) & 0 & = & u_1 & + & 0\,u_0\\
(n=3) & 2\,u_3 & = & u_2 & + & u_1\\
 & & & \vdots & & &
\end{array}$$

Evaluating the equation at $n=2$ yields $u_1=0$.
In contrast, starting from any value of u_2, one can define a solution with support $\subseteq \{2, 3,\ldots\}$.
The dimension is 1. ⊲

- For computing Laurent series solutions: while unrolling the recurrence, introduce a new indeterminate for each “free” y_ν (i.e. compute each y_n as a linear combination of $\{y_\nu : b_s(\nu-s)=0\}$), collect the linear constraints that appear, solve the resulting system (now only $r \times r$).

- For polynomial solutions:
  - (For simplicity:) Change x to $x_0 + z$ to reduce to the case of an ordinary point
  - Compute a basis of power series solutions (recurrence)
  - Search for linear combinations y of the power series solutions s.t.

$$y_{D'+1} = y_{D'+2} = \cdots = y_{D'+s} = 0$$

  (s equations, r unknowns) where $D' = \max(D, s - r - 1)$.

  Any polynomial solution is a solution of deg $\leqslant D$, so it must be found.
  Conversely, since we are at an ordinary point and $D' + s > r$, the recurrence yields $y_{D'+s+1} = 0$, etc., i.e., the solutions are polynomials.

Even better: one can combine these ideas with the fast algorithms from the other part and efficiently "jump" from one singular index to the next.

5 Rational solutions

Rational solutions reduce to polynomial solutions given a multiple of the denominator. Now assume that $y(x) \in \mathbb{K}(x)$ is a solution of $L(y) = 0$. Then:

1. any pole $\xi \in \bar{\mathbb{K}}$ of y must be a singular point of L;
2. if ξ is a pole of multiplicity m of y, by expanding y in Laurent series at ξ, we obtain a solution of y in $\mathbb{K}((x-\xi))$ of valuation $-m$ w.r.t. $x - \xi$, so $-m$ must be a root of the indicial polynomial q_ξ at $x - \xi$ of L.

Additionally, if $f \in \mathbb{K}[X]$ is the minimial polynomial of ξ, then q_ξ can be viewed as an element of $(\mathbb{K}[X] / \langle f(X) \rangle)[T]$, so its *integer* roots only depend on f (as opposed to ξ). In summary, we have the following property.

Proposition. Let $\mathscr{F}$ be the set of irreducible factors of a_r. For each $f \in \mathscr{F}$, let q_f be the "indicial polynomial at the roots of f" of L, i.e., the indicial polynomial at 0 of the operator obtained by changing x to $\xi + z$ in L where $\xi =$ class of X in $\mathbb{K}[X] / \langle f(X) \rangle$, and let

$$m_f = \max(\{m \in \mathbb{N} : q_f(-m) = 0\} \cup \{0\}).$$

Then the denominator of any (irreducible) rational solution of $L(y) = 0$ divides

$$Q = \prod_{f \in \mathscr{F}} q_f^{m_f}. \quad \triangleleft$$

This leads to the following algorithm.

Algorithm. (Rational solutions)

- Compute the q_f and Q.
- Change $y(x)$ to $w(x) / Q(x)$ in the equation.
- Compute a basis $(w_1, \ldots, w_k)$ of polynomial solutions of the resulting equation.
- Return $(w_1 / Q, \ldots, w_k / Q)$. $\triangleleft$