Exercise 1. Consider the differential equation

$$(x - 1) \, y''(x) + (-x + 3) \, y'(x) - y(x) = 0. \tag{E}$$

1. Let

$$y(x) = \sum_{n=-N}^{\infty} y_n \, (x - 1)^n$$

   be a solution of (E) and set $y_n = 0$ for $n < -N$.
   Show that the sequence $(y_n)_{n \in \mathbb{Z}}$ satisfies

$$\forall n \in \mathbb{Z}, \quad (n + 1) \, (n + 2) \, u_{n+1} = (n + 1) \, u_n.$$

2. Find all rational solutions of (E). (Do not blindly apply the full algorithm for doing so: many shortcuts are possible here.)

Exercise 2. Let $\mathbb{K}$ denote an effective field.

1. Design a divide-and-conquer algorithm that takes as input a polynomial $P(x) \in \mathbb{K}[x]$ of degree $< d$ and a point $a \in \mathbb{K}$ computes the Taylor shift $P(x + a)$ in $O(\mathrm{M}(d) \log d)$ arithmetic operations in $\mathbb{K}$.

2. Adapt the algorithm for computing $N!$ discussed in the previous lecture into one that takes as input a matrix $B(n) \in \mathbb{K}[n]_{\leqslant d}^{s \times s}$ and an integer $N \in \mathbb{N}$ and computes the product

$$B(N - 1) \cdots B(1) \, B(0)$$

in $O(s^\theta \, \mathrm{M}(\sqrt{Nd}) \log(Nd))$ operations in $\mathbb{K}$.