On the Practical Computation of Stokes Matrices

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Joint work with Michèle Loday-Richaud and Pascal Rémy

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Setting & Task

$$L = a_r(x)\frac{d^r}{dx^r} + \cdots + a_1(x)\,\frac{d}{dx} + a_0(x), \qquad a_i \in \mathbb{Q}[x]$$

$a_r(0) = 0$ — 0 singular point of **pure level 1** (def. later)

Compute the **Stokes matrices** of L at 0

- Implemented
- Fully automatic
- No genericity assumptions
- No numeric integral transforms
- Error bounds

Some Related Work

Stokes (1847) – 1970s

Mainly special equations

Sibuya, Malgrange, Écalle, Braaksma, Ramis, Loday-Richaud…
(late 1970s – early 1990s)

- Cohomological characterization
- Summability, resurgence

Thomann, Naegele, Fauvet, Richard-Jung (1990s–2000s)

Summation, Stokes matrices by classical numerical methods

van der Hoeven (2007)

Complete & fast accelero-summation algorithm

Remy (2009)

More direct algorithm for pure level 1 + other subclasses

Motivation

$$L = a_r(x)\,\frac{d^r}{dx^r} + \cdots + a_1(x)\,\frac{d}{dx} + a_0(x)$$

Applications.

- Non-integrability of Hamiltonian systems

  [Morales-Ruiz & Ramis 2001, Boucher & Weil 2003, Ramis 2024, …]

- Symbolic-numeric algorithms for exact solutions of linear differential equations.

  [van der Hoeven 2007, Llorente 2014, …]

Motivation

$$L = a_r(x)\,\frac{d^r}{dx^r} + \cdots + a_1(x)\,\frac{d}{dx} + a_0(x)$$

Theorem. [Ramis 1985]

The differential Galois group of L is the algebraic group generated by:

- the monodromy matrices,
- the exponential torus,
- the Stokes matrices

(all viewed as elements of $GL_r(\mathbb{C})$ acting on local solutions at a base point x_0).

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- Symbolic-numeric algorithms for exact solutions of linear differential equations.

  [van der Hoeven 2007, Llorente 2014, ...]

Formal Solutions

Theorem. [Fabry 1885]

The operator L has a full basis of **formal solutions**

$$\underbrace{e^{q(1/x^p)}}_{\in \bar{\mathbb{Q}}[x^{1/p}]} \overbrace{x^{\lambda}}^{\lambda \in \bar{\mathbb{Q}}[x^{1/p}]} \sum_{k=0}^{r} \underbrace{\sum_{n=0}^{\infty} c_{k,n} x^{n/p}}_{\in \bar{\mathbb{Q}}[[x^{1/p}]],\ \text{usually divergent}} \log(x)^k.$$

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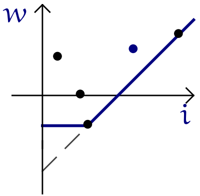
Assumption. The origin is a singular point of **pure level 1**, i.e., all exponential parts are of the form $e^{\alpha/x}$.

- This implies $p = 1$ (no ramification).
- No exp parts $\Rightarrow$ convergent series
  (regular singular point)



$$w\left(x^i \frac{d^j}{dx^j}\right) = j - i$$

Borel Summation

"Definition". A series $y(x) = x^\lambda \mathbb{C}[[x]][\log x]$ with $\mathrm{Re}(\lambda) > 0$ is **Borel-summable** in the direction θ when the following steps all make sense:

$$
\begin{array}{ccc}
y(x) \in x^\lambda \mathbb{C}[[x]][\log x] & & \\
\Big\downarrow & & \Big\uparrow \\
& \longrightarrow &
\end{array}
$$

The resulting $\mathcal{S}_\theta(y)$ is **asymptotic** to y on a domain of opening π:

$$\forall n \in \mathbb{N}, \quad \mathcal{S}_\theta(y)(x) = y_0 + y_1 x + \cdots + y_{n-1} x^{n-1} + O(x^n)$$

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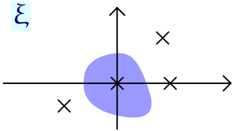

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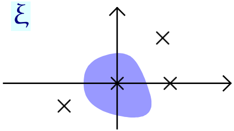
$\hat{y}(\xi)$ convergent $\xrightarrow{\text{analytic continuation}}$



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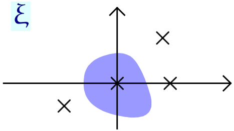
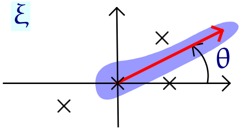
$\hat{y}(\xi)$ convergent $\xrightarrow{\text{analytic continuation}}$ $\hat{y}(\xi)$ $\Big|$ analytic on $e^{i\theta}[0,\infty)$, $e^{O(|x|)}$ at ∞





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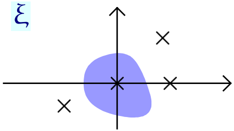
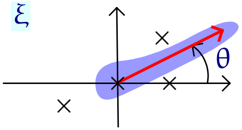
Borel: $\mathcal{B}(x^\nu \log(x)^k) = \frac{d^k}{d\nu^k} \frac{\xi^{\nu-1}}{\Gamma(\nu)}$ $\downarrow$ $\qquad$ $\uparrow$ Laplace

$\hat{y}(\xi)$ convergent $\xrightarrow{\text{analytic continuation}}$ $\hat{y}(\xi)$ $\left|\begin{array}{l}\text{analytic on } e^{i\theta}[0,\infty) \\ e^{O(|x|)} \text{ at } \infty\end{array}\right.$





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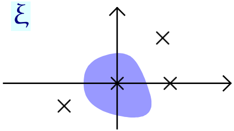

analytic continuation

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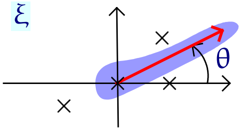
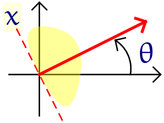


Laplace

$$\mathcal{S}_\theta(x) = \int_0^{e^{i\theta}\infty} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$



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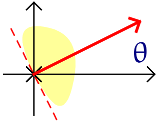
The Stokes Phenomenon in the Laplace Plane (1)
Analytic continuation of Borel sums

Proposition. Formal solutions of linear ODEs of pure level 1 are Borel-summable.

$$y(x) = e^{-\alpha/x} x^{\lambda} \sum_{k=0}^{r} \sum_{n=0}^{\infty} c_{k,n} x^n \log(x)^k \quad \longmapsto \quad S_\theta y(x)$$



$S_\theta y \sim y$ as $x \to 0$

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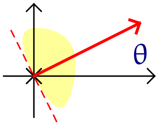

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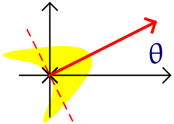
As a solution of L, the sum $\mathcal{S}_\theta(y)$ can be analytically continued around the origin



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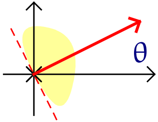

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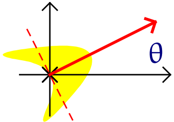


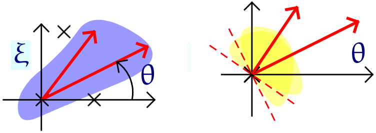
The analytic continuation remains $\sim y(x)$ in a domain of opening $> \pi$

…but the asymptotic expansion suddenly changes when crossing a **Stokes direction**

The Stokes Phenomenon in the Laplace Plane (2)

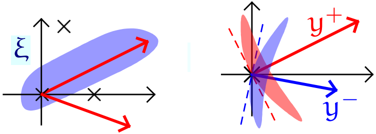
Variing the direction of summation

The sums $\mathcal{S}_\theta(y)$, $\theta \in (\theta_1, \theta_2)$ obtained by variing θ **continuously** — when possible — patch together



The sums y^+ and y^- on both sides $\omega \pm \varepsilon$ of a **singular direction** ω are (usually) different

…but have the same asymptotic expansion



($y^+ - y^-$ exponentially small in a sector of opening π)

$$y^+ \sim y^- \text{ for } |\arg x - \omega| < \frac{\pi}{2} - \varepsilon$$

Stokes Matrices

Choose a basis $Y = (y_1, \ldots, y_r)$ of formal solutions $\qquad y_i(x) = e^{\alpha_i/x} x^{\lambda_i} F_i(x, \log x)$
a singular direction ω of L

Let $Y^{\pm}$ be the sums of Y to the left/right. (Define $\mathcal{S}_\omega(e^{-\alpha/x} z(x))$ as $e^{-\alpha/x} \mathcal{S}_\omega(z(x))$.)

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Both Y^+ and Y^- are bases of analytic solutions of L on a common domain.

Definition. The **Stokes matrix** of L in the direction ω is the matrix of Y^+ in the basis Y^-:

$$Y^+ = Y^-(I + C)$$

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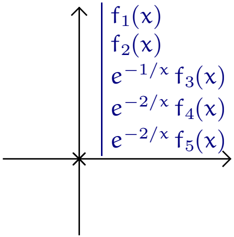
A first algorithm

- Compute the formal Borel transform $\mathcal{B}Y$ of Y
- Compute its analytic continuation to $[0, e^{i(\omega \pm \varepsilon)} \infty)$ numerically
- Compute the Laplace integrals numerically
- Compare

The Equation in the Borel Plane



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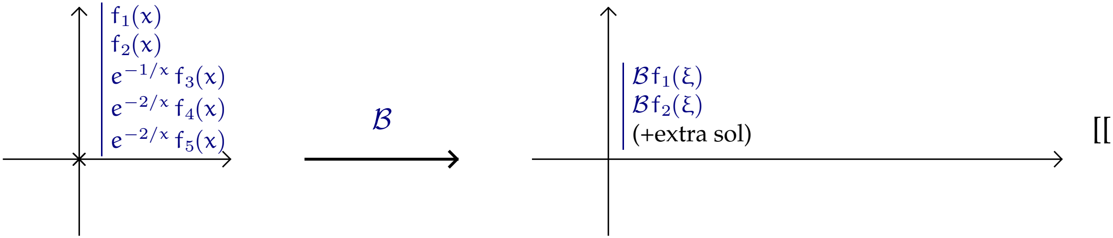

Transformed differential equation $\quad f$ solution of $L \Rightarrow \mathcal{B}f$ solution of $\hat{L}$

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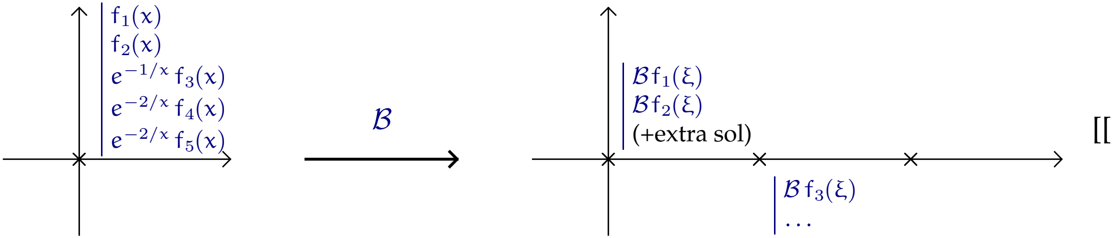

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Exponentials $\rightsquigarrow$ shifts $\quad$ $\mathcal{B}(e^{-\alpha/x} f(x)) = (\mathcal{B} f)(\xi - \alpha)$

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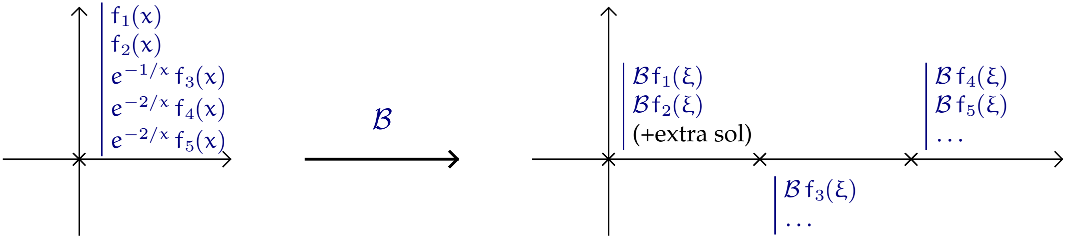

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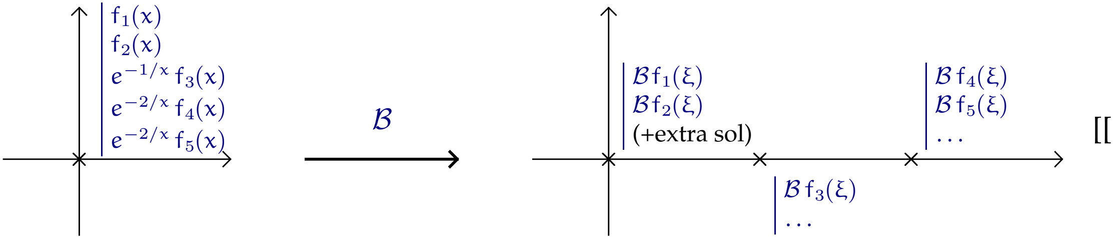

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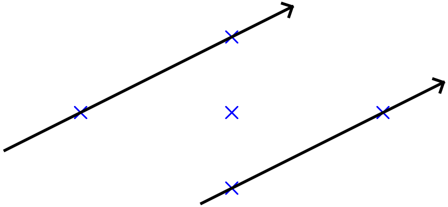
Exponentials $\rightsquigarrow$ shifts — $\mathcal{B}(e^{-\alpha/x} f(x)) = (\mathcal{B} f)(\xi - \alpha)$

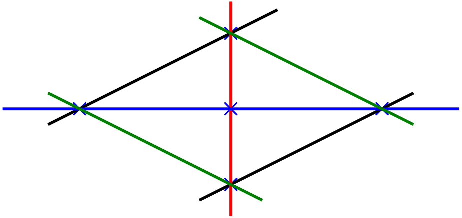
Stokes values — The finite **singular points** of $\hat{L}$ are the α such that $e^{-\alpha/x}$ is an exp part of L (incl. 0)

Regularity — For L of pure level one, these are **regular singular points**

The Stokes Phenomenon in the Borel Plane

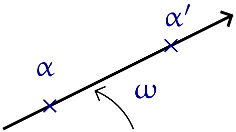
- **Singular directions** of L $=$ oriented dirs where $\hat{L}$ has $\geqslant 2$ aligned singular points





- Pair (α, α') of singular points of $\hat{L}$ $\rightsquigarrow$ block of the Stokes matrix $\omega = \arg(\alpha' - \alpha)$

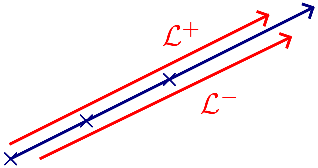
$$\begin{array}{cc} & e^{-\alpha/x} \\ & \downarrow \\ e^{-\alpha'/x} \longrightarrow & \left[\begin{array}{c|c|c} & & \\ \hline & * & \\ \hline & & \end{array}\right] \end{array}$$



To compute the column $\leftrightarrow$ basis element $y(x)$, we need to express y^+ in the basis Y^-

Contribution of a Singular Point

$$y^+ - y^- = \int_{\mathcal{L}^+} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi - \int_{\mathcal{L}^-} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$



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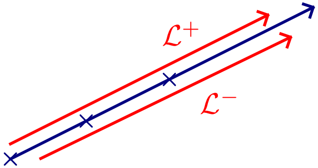
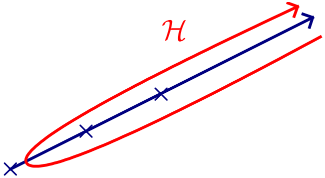


$$= \int_{\mathcal{H}} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi$$



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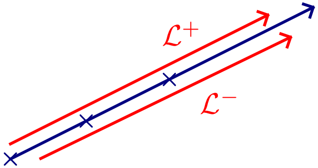

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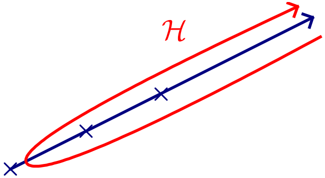
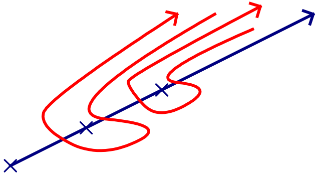


$$= \int_{\mathcal{H}_1} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi + \int_{\mathcal{H}_2} \hat{y}(\xi)\, e^{-\xi/x}\, d\xi + \cdots$$



Connection-to-Stokes Formulae

We are left with Laplace integrals on Hankel contours enclosing a single α'.

These can be computed by termwise integration of the **local expansion at α'** of $\mathcal{B}y$:

$$\mathcal{B}y(\alpha' + \zeta) \;=\; \zeta^{\lambda} \sum_{k=0}^{r} \sum_{n=0}^{\infty} c_{n,k}\, \zeta^{n} \log(\zeta)^{k}$$

$$\int_{\mathcal{H}} \mathcal{B}y(\alpha' + \zeta)\, e^{-(\alpha'+\zeta)/x}\, d\zeta \;=\;$$

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$$= \frac{d^{k}}{d\lambda^{k}} \frac{2\pi i\, (x\, e^{-\pi i})^{\lambda+n-1}}{\Gamma(-\lambda-n)}$$

$$= x^{\lambda+n-1} \times (\text{explicit polynomial in } \log(x))$$

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$$\begin{aligned}\int_{\mathcal{H}}\mathcal{B}y(\alpha'+\zeta)\,e^{-(\alpha'+\zeta)/x}\,d\zeta &= \sum_{k=0}^{r}\sum_{n=0}^{\infty} c_{n,k}\,e^{-\alpha'/x}\underbrace{\int_{\mathcal{H}}\zeta^{\lambda+n}\log(\zeta)^{k}e^{-\zeta/x}\,d\zeta}\\ &= \frac{d^{k}}{d\lambda^{k}}\,\frac{2\pi i\,(x\,e^{-\pi i})^{\lambda+n-1}}{\Gamma(-\lambda-n)}\\ &= x^{\lambda+n-1}\times(\text{explicit polynomial in }\log(x))\end{aligned}$$

- Compute enough terms of the expansion of $y^{+}-y^{-}$
- Equate the coefficients of $e^{-\alpha'/x}x^{\mu}\log(x)^{k}$ to write it in the basis Y^{-}

Summary

Naive algorithm

Stokes matrix of L in direction ω w.r.t. basis $(y_1, \dots, y_r)$:

$S := I_{r \times r}$

For $j = 1, \dots, r$

Write $y_j = e^{-\alpha/x}(\dots)$

For each singular point α' of $\hat{L}$ with $\arg(\alpha' - \alpha) = \omega$

Solve the differential equation $\hat{L}(\mathcal{B}y_j) = 0$ numerically to obtain the series expansion **at α'** of $\mathcal{B}y_j$

Deduce the coordinates of $y_j^+ - y_j$ in the basis Y^- (previous slide)

$S_{:,j} += c$

Return S

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Stokes matrix of L in direction ω w.r.t. basis $(y_1, \ldots, y_r)$:

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connection between **regular singular** points

Deduce the coordinates of $y_j^+ - y_j$ in the basis Y^- (previous slide)

$S_{:,j} \mathrel{+}= c$

Return S

Summary

Naive algorithm

Stokes matrix of L in direction ω w.r.t. basis $(y_1, \ldots, y_r)$:

$S := I_{r \times r}$

For $j = 1, \ldots, r$

Write $y_j = e^{-\alpha/x}(\ldots)$

For each singular point α' of $\hat{L}$ with $\arg(\alpha' - \alpha) = \omega$

Solve the differential equation $\hat{L}(\mathcal{B} y_j) = 0$ numerically to obtain the series expansion **at** $\boldsymbol{\alpha'}$ of $\mathcal{B} y_j$

connection between **regular singular** points

Deduce the coordinates of $y_j^+ - y_j$ in the basis Y^- (previous slide)

evaluation of $(1/\Gamma)^{(k)}$

$S_{:,j} += c$

Return S

Redundancies

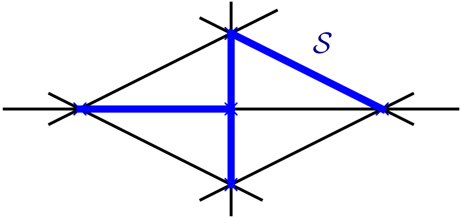
Fix for each α:
- a basis $Y_{[\alpha]}$ of the space $V_{[\alpha]}$ of formal sol. of L of exp. part $e^{-\alpha/x}$
- a basis $\hat{Y}_{[\alpha]}$ of the space $\hat{V}_{[\alpha]}$ of local sol. of $\hat{L}$ at α

Compute the matrices:

- For each α, of the map $\begin{array}{ccc} V_{[\alpha]} & \to & \hat{V}_{[\alpha]} \\ y & \mapsto & \hat{y} \end{array}$ **Borel transform matrix B_α**
- For each α', of the map $\begin{array}{ccc} \hat{V}_{[\alpha]} & \to & V_{[\alpha]} \\ \hat{y} & \mapsto & \displaystyle\int_{\mathcal{H}} \hat{y}(\zeta)\, e^{-\zeta/x}\, d\zeta \end{array}$ **Connection-to-Stokes matrix $L_{\alpha'}$**
- For each pair (α, α'), of 'the' an. cont. map $\hat{V}_{[\alpha]} \to \hat{V}_{[\alpha']}$ **Connection matrix $T_{\alpha,\alpha'}$**

The block (α, α') of the Stokes matrix in the direction $\arg(\alpha' - \alpha)$ is $L_{\alpha'}\, T_{\alpha,\alpha'}\, B_\alpha$.

Redundancies (2)



- Solve $\hat{L}$ along a spanning tree $\mathcal{S}$ of the singular points $\rightsquigarrow$ $(T_{\alpha,\alpha'})_{\alpha,\alpha' \in \mathcal{S}}$
- For other (α, α')
  - Pick known $T_{\alpha,\beta}$, $T_{\beta,\alpha'}$ s.t. the triangle (α, β, α') contains no other singular pt
  - Set $T_{\alpha,\alpha'} = \tilde{T}_{\beta,\alpha'} \tilde{T}_{\alpha,\beta}$ where $\tilde{T}_{\alpha,\beta} = T_{\alpha,\beta}$ up to a branch correction

Conclusion

Stokes matrices of LODE of pure level 1 are computable in practice

$\longrightarrow$ code available

$\longrightarrow$ roughly as fast as regular singular connection

$\longrightarrow$ rigorous error bounds

Reuse parts of this for equations with multiple levels?