Numerical evaluation of D-finite functions in `ore_algebra`

A progress report

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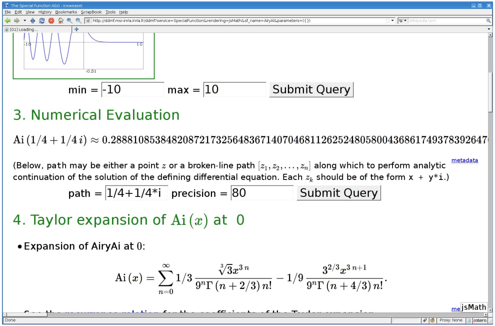
arxiv:1607.01967 [cs.SC]

1 What it *is*

A Better NumGfun



http://ddmf.msr-inria.inria.fr

[Benoit, Chyzak, Darrasse, Gerhold, Grégoire, Koutschan, M., Salvy 2010–]

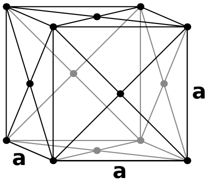
NumGfun [M. 2010]

- General D-finite functions
- Arbitrary precision
- Rigorous error bounds

- Maple
- Oriented towards special functions

Airy function: $\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$

dop6 $= 410085196915322880\, z^{35} + 1129052664742115663520\, z^{34} + 1171669263761496\backslash$
$1489920\, z^{33} + 690817401287078917363200\, z^{32} + 27204862643846611522761600\, z^{31} + 7788\backslash$
$1140691824722861849 7600\, z^{30} + 1704438412411524078142979 2800\, z^{29} + 29424523406685\backslash$
$0000428339092800\, z^{28} + 40834245878051170602724761258 00\, z^{27} + 4597302954911979623\backslash$
$5386142827300\, z^{26} + 419695598890898253203455876749930\, z^{25} + 3064297176174091671 7\backslash$
$17985958725620\, z^{24} + 1716958448925969638875580463603 3570\, z^{23} + 6458177196168481 0\backslash$
$077279475394020500\, z^{22} + 517142219342720994204761262167 66700\, z^{21} - 1473967391504\backslash$
$437899277380487903179960\, z^{20} - 142375543413213353353920231928 72385940\, z^{19} - 8321\backslash$
$634013439311501683422098038 4454340\, z^{18} - 364019154328107562568847906822 48806\backslash$
$3550\, z^{17} - 126157147851340108817703509327 5526304300\, z^{16} - 3528341032098896995323\backslash$
$439017117956856150\, z^{15} - 796436951859377802952105607044279 4466900\, z^{14} - 14280500\backslash$
$7261627862547128411638750017 28600\, z^{13} - 19534653115686342543580831960941 97891\backslash$
$8000\, z^{12} - 1839878333422238008423801242870473 1960000\, z^{11} - 7553741785990309 35723\backslash$
$405478617748800000 0\, z^{10} + 888743230941952240398397617177569 7600000\, z^{9} + 21137039\backslash$
$158366320685856256980012112000000\, z^{8} + 2268269355393480\backslash$
$469044664729550880000000 0\, z^{7} + 14938183428146261190546 3546\backslash$
$7161600000000 0\, z^{6} + 469024652858481632994019940044800000\backslash$
$0000\, z^{5} - 872829008634785738926162452480000000000\, z^{4} - 11043\backslash$
$279407797458901507731456000000000 00\, z^{3} - 3538987082075808 5\backslash$
$677291929600000000000 0\, z^{2} - 5207934291077444487413760000 00\backslash$
$00000000\, z - 242879062193188503552000000000000000 0 +$



Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{aligned}
&(396415690351478784 0\,z^{36} + 1104718489963413534720\,z^{35} + \\
&1178710887399303528345 60\,z^{34} + 71832875166444796157952 00\,z^{33} + 29310583\backslash \\
&5218942903781855360\,z^{32} + 870657237873498477679950240 0\,z^{31} + 1979497761381158661\backslash \\
&33849254880\,z^{30} + 3555494624146318548046453851120\,z^{29} + 514578986720138650981112\backslash \\
&91247320\,z^{28} + 60652283497953184068452162523702 0\,z^{27} + 5835366836846027182876920\backslash \\
&856348950\,z^{26} + 45455202501826358974606219981974015\,z^{25} + 2791534044675020629485\backslash \\
&31557838260750\,z^{24} + 12522753993728371340615070426289087 95\,z^{23} + 294318280292355\backslash \\
&2038161307584706940070\,z^{22} - 10483513115206289398510413216920199750\,z^{21} - 169948\backslash \\
&1829335074791612575655686165307 00\,z^{20} - 1154969594776277160649077785983820 55\backslash \\
&3870\,z^{19} - 5548694490781020038019823355124585193590\,z^{18} - 207452295175774512723 7\backslash \\
&715824197091543924 5\,z^{17} - 62232963928794638659423069651761724690290\,z^{16} - 150810\backslash \\
&0459019789328641634930464054612621 05\,z^{15} - 2925286265230056616293902368830468 59\backslash \\
&976150\,z^{14} - 44139509606318314883900817224858033778030 0\,z^{13} - 484123578764537043\backslash \\
&031861206473715269343000\,z^{12} - 31297658464933476345181066385800419642000 0\,z^{11} + \\
&3141513349990995023483191539586929360000 0\,z^{10} + 330738460087674555468491482629\backslash \\
&558468000000\,z^{9} + 391096978918364972225128472061480072000000\,z^{8} + 23246017042594\backslash \\
&802734543485030527952000000 0\,z^{7} + 1806013493488429983484709934556800000000 0\,z^{6} - \\
&10021340089119210237029332603699200000000 0\,z^{5} - 838590645159851364959030994585 6\backslash \\
&0000000000\,z^{4} - 29487051768804049058487328972800000000000\,z^{3} - 551648271974304 22\backslash \\
&28472840192000000000000\,z^{2} - 379553789663505987184558080000000000000\,z + 8379327\backslash \\
&645665003372544000000000000000)\,\mathrm{Dz} +
\end{aligned}
$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$(8133356405487237120\,z^{37} + 229413178204366431744 0\,z^{36} +$
$25129532853476219363328 0\,z^{35} + 1579545301524081697009152 0\,z^{34} + 666093618246\backslash$
$765502077439680\,z^{33} + 204693757124168430409093761 60\,z^{32} + 4818173382673386397833\backslash$
$28749120\,z^{31} + 896797312621202003251799121696 0\,z^{30} + 1346969148543045367222818 66\backslash$
$954300\,z^{29} + 165183365498487607982012588567865 0\,z^{28} + 1660749002634342953281157 5\backslash$
$311949230\,z^{27} + 136263869454304799146859253346813455\,z^{26} + 8958653193274714476 38\backslash$
$111289873238710\,z^{25} + 448971107426438490652992599025479326 5\,z^{24} + 14491852283494\backslash$
$5778266540039325477116 90\,z^{23} - 1685090664711945469831746481335427 50\,z^{22} - 386265\backslash$
$894549826881229123104470731096440\,z^{21} - 316325913106056858454611334378198756\backslash$
$1220\,z^{20} - 166361820694138211705446840475562205681 50\,z^{19} - 662467400893936760809\backslash$
$815371303780906585 25\,z^{18} - 2090802458728506315663121374496195615437 30\,z^{17} -$
$529097465740104776391772834675033946593335\,z^{16} - 106503862075731392939163936 10\backslash$
$32363453750930\,z^{15} - 165365164468562014216700942212402255522170 0\,z^{14} - 182938347\backslash$
$4513975929874027770563298831967800\,z^{13} - 1088709896836905806660284149277621568 3\backslash$
$28000\,z^{12} + 437384067337328886944483963336952904080000\,z^{11} + 1678380365365432006\backslash$
$62545147323626901200000 0\,z^{10} + 15643853555920279359226831628986551120000 00\,z^{9} +$
$303607398715325954207032303663107840000000\,z^{8} - 929554263384554771136483801\backslash$
$584745600000000\,z^{7} - 1334658535726482371536908049179648000000000\,z^{6} - 9419775340\backslash$
$0652483718287926356480000000000 0\,z^{5} - 350297653852787677589275016601600000 00\backslash$
$0000\,z^{4} - 642862414738921482346405847040000000000000\,z^{3} - 3745850575154857509858 5\backslash$
$088000000000000 00\,z^{2} + 668706777983396247404544000000000000000000\,z + 7286371865795\backslash$
$655106560000000000000000 00)\,\mathrm{D}\mathrm{z}^{2} +$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{aligned}
&(6219625486549063680\,z^{38} + 177553133630802252880\,z^{37} + \\
&1994099966351325897523200\,z^{36} + 1290486249759244892016384 0\,z^{35} + 561222248755\backslash \\
&128125708191680\,z^{34} + 177986954210726976694687396 80\,z^{33} + 432530168604725658189 2\backslash \\
&10596640\,z^{32} + 8315189920333341531658617695280\,z^{31} + 1291037239045957714099282 32\backslash \\
&487740\,z^{30} + 16391907385319861706996470978037 90\,z^{29} + 171110407098408230357607 57\backslash \\
&618682440\,z^{28} + 146518901443861803658771329866897880\,z^{27} + 101553427880666984315\backslash \\
&9745327151252620\,z^{26} + 54963382760530760750687544673101027 60\,z^{25} + 2089057420971\backslash \\
&4927539267068315744951640\,z^{24} + 301646099705919471898270762349220070 50\,z^{23} - \\
&28912741628152937614209563101551926712 0\,z^{22} - 30536692798730633467931505919379\backslash \\
&74700130\,z^{21} - 17566486109105161467894504789161406270600\,z^{20} - 73673650638461574\backslash \\
&5386797430970500511152 20\,z^{19} - 2404074389675573989133172969753365747029 80\,z^{18} - \\
&61916829368763951125106727397502019711410 0\,z^{17} - 12412253032904606237959908599\backslash \\
&05959226579320\,z^{16} - 1835553795134837646262350779261931882894750\,z^{15} - 167031460\backslash \\
&28371411108456407060310120735557 00\,z^{14} + 306687486186229618627508760752757 1093\backslash \\
&2000\,z^{13} + 29315941553133903289357161870016145682600 00\,z^{12} + 491945645889966649 8\backslash \\
&68406970838854828560 0000\,z^{11} + 377736564624376265310479588420614333200 0000\,z^{10} + \\
&68195639154415674514017863641593600000000\,z^{9} - 311881093752325372623566678 26\backslash \\
&660968 00000000\,z^{8} - 377183378739961670425890880829428800000000 0\,z^{7} - 24884448319\backslash \\
&3099682490895498914432000000000 0\,z^{6} - 893959103422093803323305262745600000 00\backslash \\
&0000\,z^{5} - 100083705719332806676962561024000000000000 0\,z^{4} + 2564203084567737476418\backslash \\
&0172800000000000000\,z^{3} + 10280761031833373040014131200000000000000 00\,z^{2} + 10337540\backslash \\
&0845975856824320000000000000000000\,z)\,\mathrm{Dz}^{3} +
\end{aligned}
$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$(2192816677949990400\,z^{39} + 63349021347730876800 0\,z^{38} +$$
$$72864986011484455353600\,z^{37} + 484748686979553726053280 0\,z^{36} + 21701401\backslash$$
$$7048761645614816000\,z^{35} + 70880169958001247079960905 60\,z^{34} + 177409657131610270 4\backslash$$
$$82016190640\,z^{33} + 35130899127365841565492386766 20\,z^{32} + 562015877327404499596706\backslash$$
$$75451690\,z^{31} + 735847759326730529024504240988015\,z^{30} + 793441106307331443298848 2\backslash$$
$$485900500\,z^{29} + 7040517191263028635994557189677411 0\,z^{28} + 508882813920850610699 2\backslash$$
$$35633677324220\,z^{27} + 291338677229064650128281265554601147 5\,z^{26} + 122379897749644\backslash$$
$$633850628909269632159 50\,z^{25} + 273031638745556160521558984755243862 10\,z^{24} - 84400\backslash$$
$$7242726014050652717732643972095 30\,z^{23} - 1309329548085768562973129072537724 16\backslash$$
$$4955\,z^{22} - 822926919906244244426426023497784780536 0\,z^{21} - 359487409188444751403 1\backslash$$
$$857484000111521367 0\,z^{20} - 119246681320333134593914488403328142970080\,z^{19} - 30427\backslash$$
$$330829743863016283709904128505054645 5\,z^{18} - 577145030895901790697311126386896 03\backslash$$
$$6767490\,z^{17} - 7077781677901366027280381449676709168373 50\,z^{16} - 153341406553907 33\backslash$$
$$4245470038125935935813900\,z^{15} + 16112029206288259429402194065158764195420 00\,z^{14} +$$
$$418899361620501704689973912421154454346000 0\,z^{13} + 569962639208201803745325939 6\backslash$$
$$194906388000000\,z^{12} + 41451401872033098361833113982524699640000 00\,z^{11} + 95068892\backslash$$
$$3971337731996303622505069600000 00\,z^{10} - 322447694713681036240924354097202960000\backslash$$
$$0000\,z^9 - 363683552813892830276766498739953600000000 0\,z^8 - 2103918711236593285 37\backslash$$
$$31961115328000000000 0\,z^7 - 5515906024254143841641178582528000000000 0\,z^6 + 1162\backslash$$
$$159256944109024208981780480000000000 00\,z^5 + 1128385276844823710179815628800000 0\backslash$$
$$0000000\,z^4 + 266404409116908139735547904000000000000000\,z^3 + 242727262779317760 73\backslash$$
$$728000000000000000000\,z^2)\,\mathrm{Dz}^4 +$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$(390720062616543744\,z^{40} + 1142166614243603074560\,z^{39} +$$
$$1344082235161596306969 6\,z^{38} + 917965180366474611870720\,z^{37} + 422376739322\backslash$$
$$637759885705 60\,z^{36} + 1418218839310481932976078400\,z^{35} + 36487206836408001197689 9\backslash$$
$$10640\,z^{34} + 74250415206276560223775945272 0\,z^{33} + 1220569466691901164246256065\backslash$$
$$0930\,z^{32} + 1642510223944437789867637365394 05\,z^{31} + 1821767115612836434053737404 7\backslash$$
$$55054\,z^{30} + 1665793830282400726724372436543419 1\,z^{29} + 1245334608496202000097114 4\backslash$$
$$5328730256\,z^{28} + 7435937964429085400705322454883782 05\,z^{27} + 33360046070881075316\backslash$$
$$34361889061221370\,z^{26} + 90202221618547360842026298243905470 47\,z^{25} - 919882472294\backslash$$
$$3404205447421299404277112\,z^{24} - 2794498688024025141756770417894925700 17\,z^{23} -$$
$$1907863427661939885576723126598906643790\,z^{22} - 857408364647571005075756554267 2\backslash$$
$$979674555\,z^{21} - 2840558729684723107018360685658377081172 0\,z^{20} - 6957425817531295\backslash$$
$$5514440713973653616428745\,z^{19} - 1149914368924877116699378498249125174303 30\,z^{18} -$$
$$70378017201579863364495432167182725333675\,z^{17} + 2616419665010891478436560832 16\backslash$$
$$157842879550\,z^{16} + 1049410824795136384837467209810025539400000\,z^{15} + 20896637809\backslash$$
$$6427299760015989881180051339 0000\,z^{14} + 2597860026796363803313313450929745040 00\backslash$$
$$0000\,z^{13} + 1759136834585156085432113720072647266000000\,z^{12} - 1346756149912237131\backslash$$
$$0874092829081128000000 0\,z^{11} - 1578996098791370746284707453439169200000000\,z^{10} -$$
$$1556811681322720025894531955998040000000000 0\,z^{9} - 6789089176134994417613424347 9\backslash$$
$$5200000000000\,z^{8} + 314922159293504699010388187520000000000 0\,z^{7} + 192407344459752\backslash$$
$$425261121833472000000000000\,z^{6} + 10412684021044478206062895104000000000000 0\,z^{5} +$$
$$209822766430453935990128640000000000000000\,z^{4} + 178743809832799664332800000000000\backslash$$
$$00000000\,z^{3})\,\mathrm{Dz}^{5} +$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$(35882454730090752\,z^{41} + 1061260405161448665 6\,z^{40} + 1276532600942212775168\,z^{39} +$$

$$8939398012943303209632 0\,z^{38} + 42216068389834732281970 08\,z^{37} + 145494567985766\backslash$$

$$484898923048\,z^{36} + 384082800449092006095096948 0\,z^{35} + 801600623882677271722119 8\backslash$$

$$5080\,z^{34} + 135085509439800690268287092205 0\,z^{33} + 186310828926305368242229494 0\backslash$$

$$9585\,z^{32} + 2118157968344640547119736453221 42\,z^{31} + 1986708322085667572665525016 0\backslash$$

$$37411\,z^{30} + 1526308238303140677042902275876204 8\,z^{29} + 940687328520892057561307 73\backslash$$

$$605094705\,z^{28} + 4410553762290959215133571309188113 38\,z^{27} + 131963694549876126497\backslash$$

$$3744224282378779\,z^{26} - 1376268096732267953995912640790411 12\,z^{25} - 31072001737970\backslash$$

$$299221405533198706303141\,z^{24} - 2268861766669185609872402007686316931 50\,z^{23} -$$

$$1033954017266382248984767586852072344191\,z^{22} - 33567329462243736016490879373 4\backslash$$

$$9109785896\,z^{21} - 7573126212785007618891225542456994124245\,z^{20} - 9076459539413303\backslash$$

$$18464172213477657389581 0\,z^{19} + 1027867124809033537740891835881540878842 5\,z^{18} +$$

$$8514927435704329238592503365329429185355 0\,z^{17} + 2406893603584982960079390961\backslash$$

$$87740586134000\,z^{16} + 4294098789219576487905557752682427433500 00\,z^{15} + 4957792250\backslash$$

$$4677190642025554034828134480000 0\,z^{14} + 287121363379312616871562346484465378 00\backslash$$

$$0000\,z^{13} - 1196826520075483509544578567502507200000 00\,z^{12} - 3956834655926808674 0\backslash$$

$$1293480616198000000000\,z^{11} - 3273834627550423859497476912408240000000 00\,z^{10} -$$

$$866425754505013910667872020195200000000 00\,z^{9} + 597046839721706795489319772224 00\backslash$$

$$000000000\,z^{8} + 7251161027741239099083936307200000000000 00\,z^{7} + 33882896755872071 9\backslash$$

$$5688626176000000000000 00\,z^{6} + 6311156771304917325766656000000000000000 00\,z^{5} + 51232\backslash$$

$$3021813756999680000000000000000000 0\,z^{4})\,\mathrm{Dz}^{6} +$$

Face-Centered Cubic Lattices

$(1600200173148416\, z^{42} + 478782978712278912\, z^{41} + 58815380786135567104\, z^{40} +$
$4218590040421804170816\, z^{39} + 204216444469816446653424\, z^{38} + 7214118\backslash$
$6241199375295411 60\, z^{37} + 195106070712453547506798168\, z^{36} + 41688703195243681975 3\backslash$
$3959000\, z^{35} + 718744127953129405117956689 40\, z^{34} + 1013528039913249207367842 37\backslash$
$8270\, z^{33} + 117759241810488938483573956706 76\, z^{32} + 112862055818213392356279768 22\backslash$
$5402\, z^{31} + 886340494836475569866741139358344\, z^{30} + 55926759731865674372797336853\backslash$
$51646\, z^{29} + 26983635759333711243427828354079724\, z^{28} + 850593884632641423136625 26\backslash$
$542420618\, z^{27} + 2481695683318164448040331315073 5864\, z^{26} - 173973152992350329598 4\backslash$
$796806526752758\, z^{25} - 13215685421423157401833903137021991092\, z^{24} - 6010151473251\backslash$
$77793295427498988934538 58\, z^{23} - 187406121933740017212741167478185137320\, z^{22} -$
$3670887867367150639084124621661565155 66\, z^{21} - 13633123830398834900141541 81\backslash$
$532146340\, z^{20} + 20529376322297997536667585043036814461 50\, z^{19} + 89422208647113020\backslash$
$920239501683485348563 00\, z^{18} + 2211277908345604739979169031967335680 8000\, z^{17} +$
$36662299830964853548300895468723502480000\, z^{16} + 38663936209054739955701649076 78\backslash$
$4708400000\, z^{15} + 15575841209632684184725074680551176000000\, z^{14} - 233997759271107\backslash$
$78754739301057544560000000\, z^{13} - 45957581844555068108338692961807200000000\, z^{12} -$
$32525005285459811112066289505232000000000\, z^{11} - 3661218292337523929046664304\backslash$
$640000000000\, z^{10} + 1097039530150661129281453716480000000000 0\, z^{9} + 97131124051979\backslash$
$35942595533824000000000000\, z^{8} + 4260978719420783892503777280000000000000\, z^{7} +$
$7564670791225350266880000000000000000000\, z^{6} + 59201771409589697740800000000000000\backslash$
$0000\, z^{5})\, \mathrm{Dz}^{7} +$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{array}{l}
(27122036833024\,z^{43} + 8208413201024064\,z^{42} + 1028987679702510976\,z^{41} + 7551845\backslash \\
1137118783792\,z^{40} + 374319561938198990 7184\,z^{39} + 13536963807754693626 1428\,z^{38} + \\
3745615314367420203992832\,z^{37} + 81811619367860049045984675\,z^{36} + 1440466637248\backslash \\
203913774334250\,z^{35} + 2072433111304027502371917285 0\,z^{34} + 2454466275416520460977\backslash \\
92768214\,z^{33} + 2395828801191215780780578117794\,z^{32} + 19147407470673111231862 2494\backslash \\
18166\,z^{31} + 1228639636214967463701886596967 02\,z^{30} + 60262125564848592437870067 23\backslash \\
31054\,z^{29} + 193519266430161747613733767108836 0\,z^{28} + 69415271203678326424364429 0\backslash \\
673234\,z^{27} - 3903004288581893545590128913387262 2\,z^{26} - 2976459628039331965648737\backslash \\
33670191774\,z^{25} - 1329742929728007215704002549281591538\,z^{24} - 390398961482564881\backslash \\
922443265720872764 6\,z^{23} - 6038015534019664017777438417359311914\,z^{22} + 7565280951\backslash \\
15600975099282347955069417 0\,z^{21} + 8332812633696018310177123954988378632 5\,z^{20} + \\
29785983697247118038201732716290595590 0\,z^{19} + 68122669439368525201713007 3908\backslash \\
325840500\,z^{18} + 10555043169325862266133900443100179200 00\,z^{17} + 9749826251441106 5\backslash \\
4834660688990434600000\,z^{16} + 80921481727424794623135930623472000000\,z^{15} - 124569\backslash \\
2778975371208980497936649580000000\,z^{14} - 18776129721660465428415258915480000 0\backslash \\
0000\,z^{13} - 1186201691981014544058180007080000000000\,z^{12} - 4424139263701398029187\backslash \\
830400000000000\,z^{11} + 5265884988550235591341773120000000000000\,z^{10} + 410999234738\backslash \\
834010247469568000000000000\,z^{9} + 1743330122138109580511846400000000000000\,z^{8} + \\
2995053035474379315363840000000000000000\,z^{7} + 22769912080611422208000000000000000\backslash \\
0000\,z^{6})\,\mathrm{Dz}^{8}
\end{array}
$$

order 8, degree 43, 43-digit coefficients

Code Generation for Special Functions

[Lauter-M. 2014–]

$$x^2 Y_1'' + x Y_1' + (x^2 - 1) Y_1 = 0$$
$$Y_1(x) \sim -\frac{2}{\pi x} + \frac{x \ln x}{\pi} + \cdots$$
as $x \to 0$

?

```
double BesselY1 (double x) {
    // generated code
}
```

Input: differential equation + initial values defining f
domain $[a, b]$
target accuracy
(+ processor, floating-point format, ...)

Output: C code

Spec: $\left| \frac{\texttt{implem}(x) - f(x)}{f(x)} \right| \leqslant \varepsilon$ for all $x \in [a, b] \cap$ `double`

Sagenstein: A Prototype Code Generator

differential equation

↓

Frontend

truncated Taylor series

↓

compact rough approximation

↓

Backend

minimax approximation

↓

representable approximation

↓

implemented polynomial

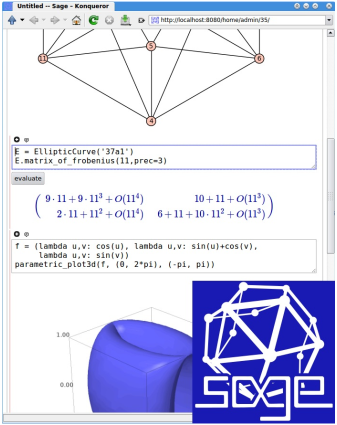
↓

C code

- ”Rigorous **polynomial approximation** pipeline”
- **Frontend** ≈ ore_algebra

https://scm.gforge.inria.fr/anonscm/git/metalibm/sagenstein.git

SageMath



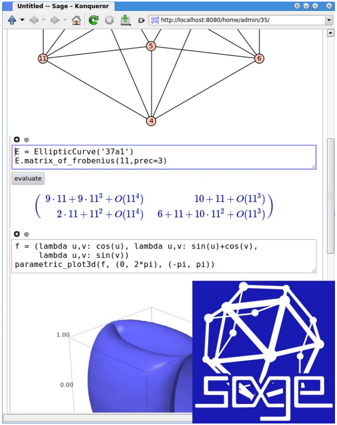
- Python library
- "A viable alternative to Magma, Maple, Mathematica and Matlab"

```
sage: Pols.<z> = PolynomialRing(QQ)
sage: (z + 1)*(z-1)
```

http://sagemath.org/

GNU GPL v2+

SageMath



- Python library
- "A viable alternative to Magma, Maple, Mathematica and Matlab"

```
sage: Pols.<z> = PolynomialRing(QQ)
sage: (z + 1)*(z-1)
z^2 - 1
```

http://sagemath.org/

GNU GPL v2+

ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

```
sage: from ore_algebra import OreAlgebra
sage: DiffOps.<Dz> = OreAlgebra(Pols)
sage: DiffOps
sage: Dz*z
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

http://kauers.de/software.html

GNU GPL v2+

ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

```
sage: from ore_algebra import OreAlgebra
sage: DiffOps.<Dz> = OreAlgebra(Pols)
sage: DiffOps
Univariate Ore algebra in Dz over Univariate Polynomial Ring in z
over Rational Field
sage: Dz*z
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

http://kauers.de/software.html

GNU GPL v2+

ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

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Univariate Ore algebra in Dz over Univariate Polynomial Ring in z
over Rational Field

sage: Dz*z

z*Dz + 1
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing…

http://kauers.de/software.html

GNU GPL v2+

The analytic Branch

- **Symbolic-numeric** extensions for ore_algebra
- Both for “end users” and for prototyping algorithms
- Fixes & some generally useful code go directly into Sage

- Part of the official release of ore_algebra since v0.3

  `http://kauers.de/software.html`

- Latest development version + more info

  `http://marc.mezzarobba.net/code/ore_algebra-analytic`

2 What it *does*

D-Finite Functions

An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

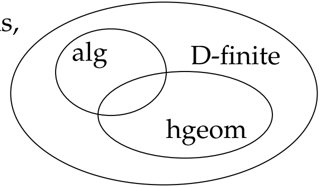
$$a_r(z)\,y^{(r)}(z) + \cdots + a_1(z)y'(z) + a_0(z)\,y(z) = 0, \qquad a_j \in \mathbb{C}[z]$$

i.e. $$(a_r(z)\,D_z^r + \cdots + a_1(z)\,D_z + a_0(z))(y) = 0.$$

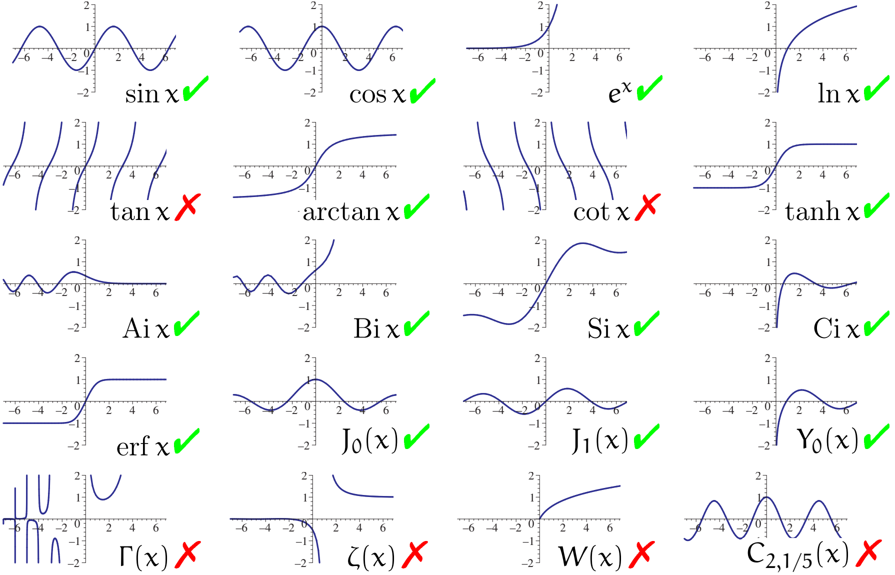
Philosophy:

[Lánczos 1956; Stanley, Zeilberger... 1980–]

Provide **general algorithms** for D-finite functions, using { ODE + initial values } as a data structure.



D-Finite Functions



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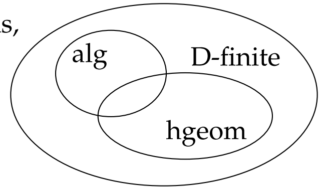

D-Finite Functions

An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

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i.e. $$(a_r(z)\, D_z^r + \cdots + a_1(z)\, D_z + a_0(z))(y) = 0.$$

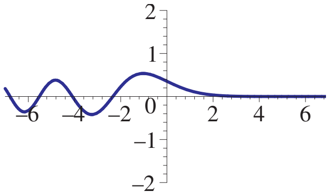
Existence Theorem [Cauchy]

Let $U \subseteq \mathbb{C}$ be a simply connected domain where $a_r(z) \neq 0$.

Then $a_r(z)\, D_z^r + \cdots + a_0(z)$ admits an r-dimensional vector space of solutions analytic on U.

A solution is characterized by the **initial values** $y(z_0), y'(z_0), \ldots, y^{(r-1)}(z_0)$ for any $z_0 \in U$.

Special Functions



$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$$

$$\mathrm{Ai}(0) = \Gamma(2/3)^{-1}\,3^{-3/2}$$

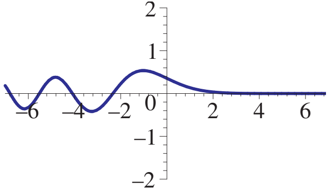
$$\mathrm{Ai}'(0) = -\Gamma(1/3)^{-1}\,3^{-1/2}$$

```
sage: diffop = Dz^2 - z

sage: diffop.numerical_solution(
       [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
       [0, i], 1e-40)

sage: ComplexBallField(138)(i).airy_ai()
```

Special Functions



$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$$

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```
sage: diffop = Dz^2 - z

sage: diffop.numerical_solution(
       [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
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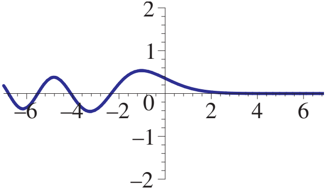
[0.3314933054321411889845293326171343458866 +/- 5.51e-41]
+ [-0.31744985896844377347764292790925852645896 +/- 7.22e-42]*I

sage: ComplexBallField(138)(i).airy_ai()
```

Special Functions



$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$$

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+ [-0.31744985896844377347764292790925852645896 +/- 7.22e-42]*I

sage: ComplexBallField(138)(i).airy_ai()

[0.33149330543214118898452933261713434588655 +/- 5.25e-42]
+ [-0.31744985896844377347764292790925852645896 +/- 1.59e-42]*I
```

Polynomial Approximations

```
sage: diffop

sage: from ore_algebra.analytic import polynomial_approximation as
      polapprox

sage: polapprox.on_interval(diffop,
        [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
        [[-1,1]], 1e-6)
```

Polynomial Approximations

```
sage: diffop

Dz^2 - z

sage: from ore_algebra.analytic import polynomial_approximation as
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sage: polapprox.on_interval(diffop,
        [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
        [[-1,1]], 1e-6)
```

Polynomial Approximations

```
sage: diffop
Dz^2 - z
sage: from ore_algebra.analytic import polynomial_approximation as
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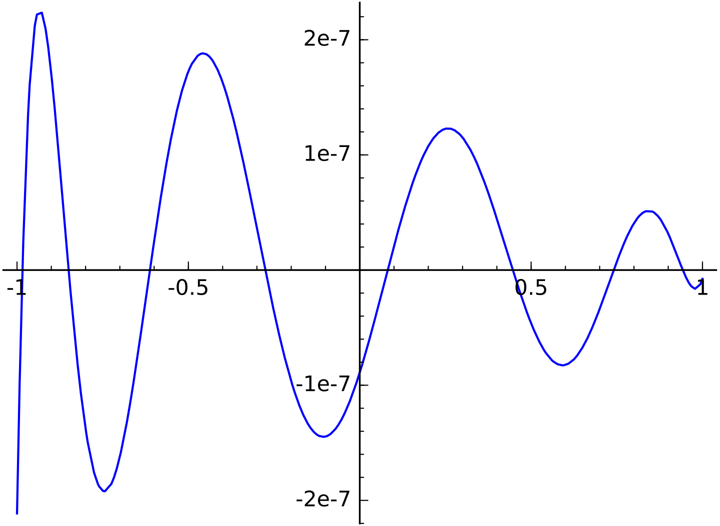
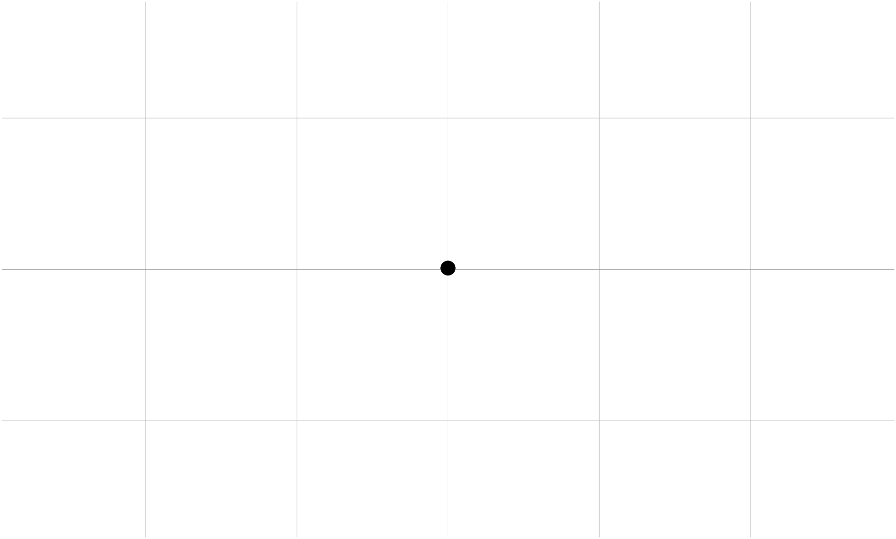
sage: polapprox.on_interval(diffop,
        [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
        [[-1,1]], 1e-6)
[-0.0004520571868735569 +/- 3.87e-20]*z^7 + [0.001957043604661954
+/- 8.46e-20]*z^6 + [-4.604372905512275 9e-5 +/- 9.82e-22]*z^5 + [-
0.02155557873905227 1 +/- 4.65e-19]*z^4 + [0.05918412106872052 2 +/-
6.71e-19]*z^3 + [-2.848455534105943 0e-6 +/- 7.52e-23]*z^2 + [-
0.2588203555063016 +/- 5.02e-17]*z + [0.355028 +/- 4.87e-7]
```

Polynomial Approximations



Analytic Continuation

$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad [y(z)=\arctan(z)]$



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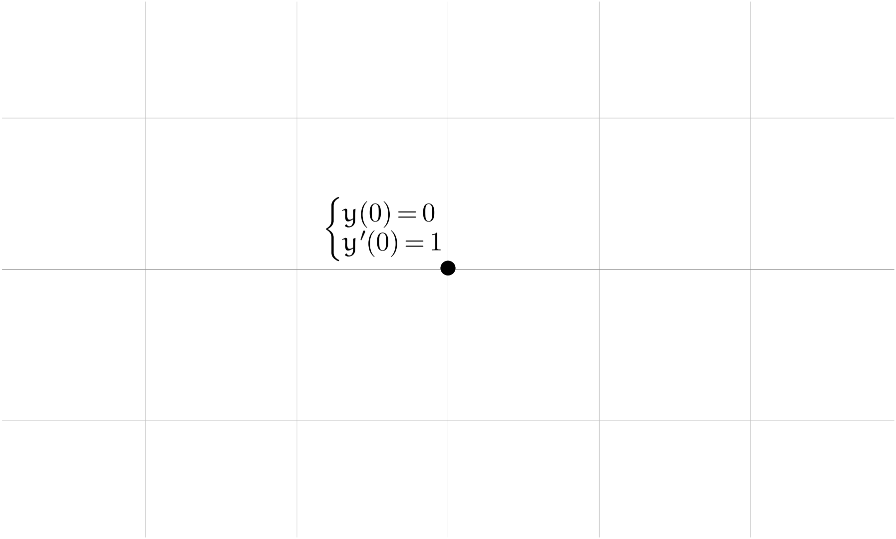

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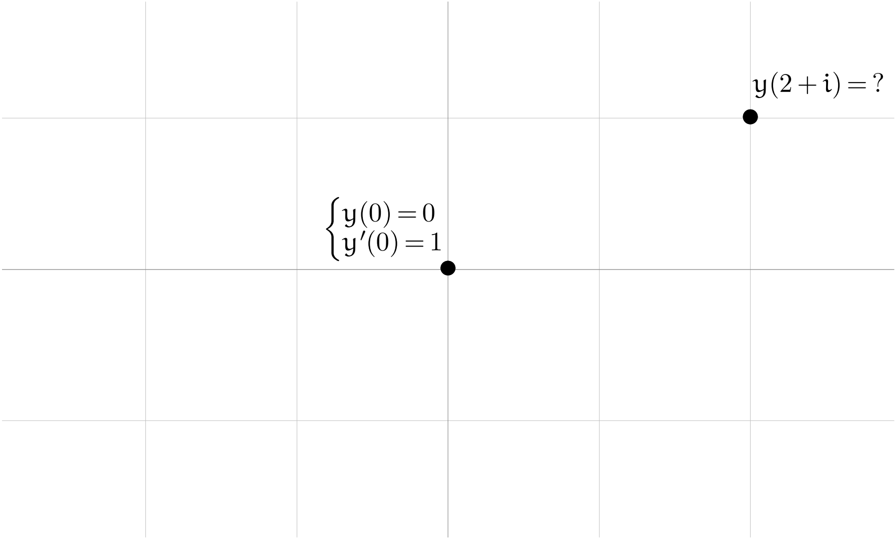

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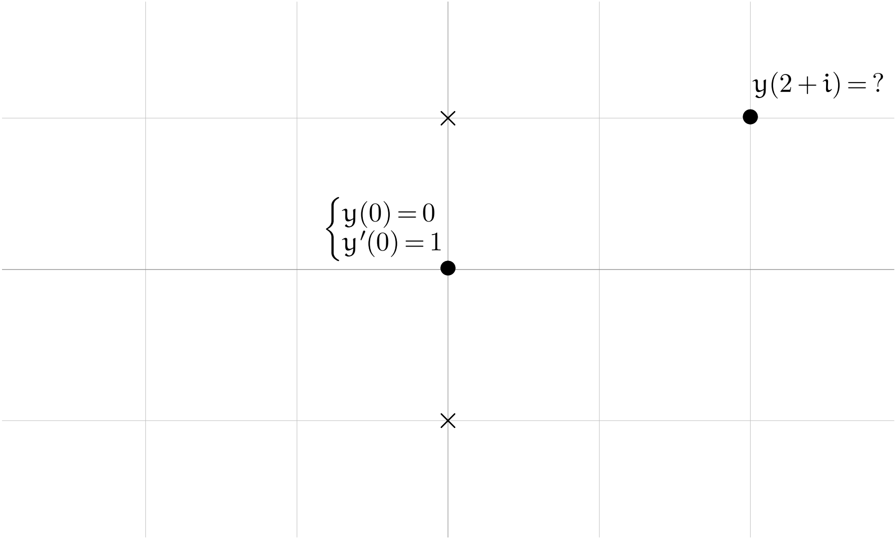

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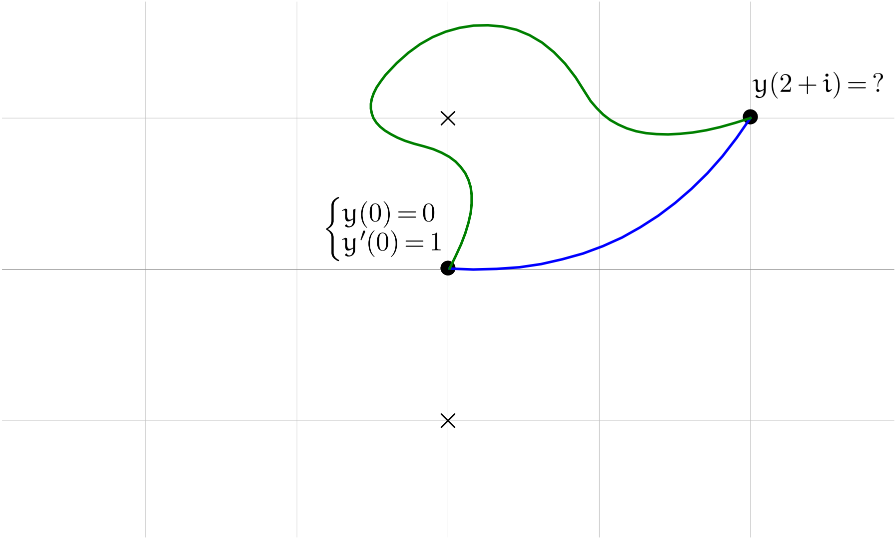

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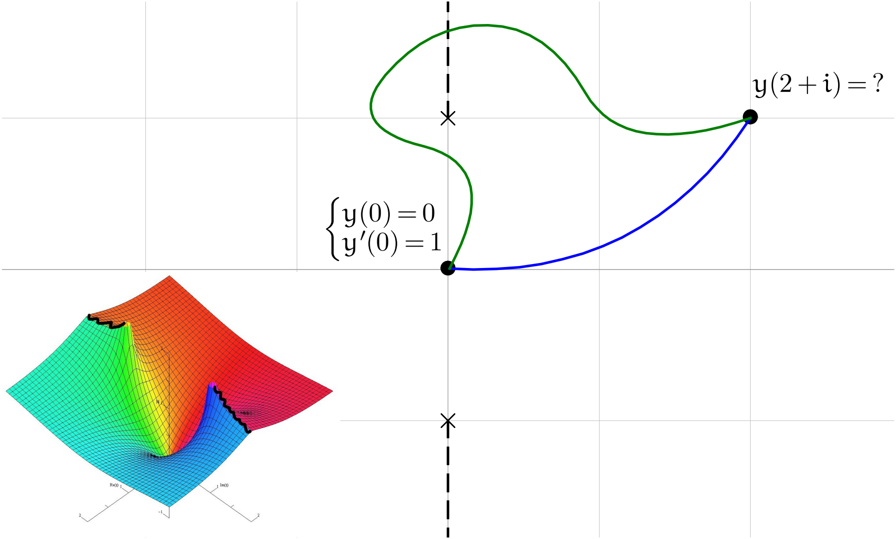

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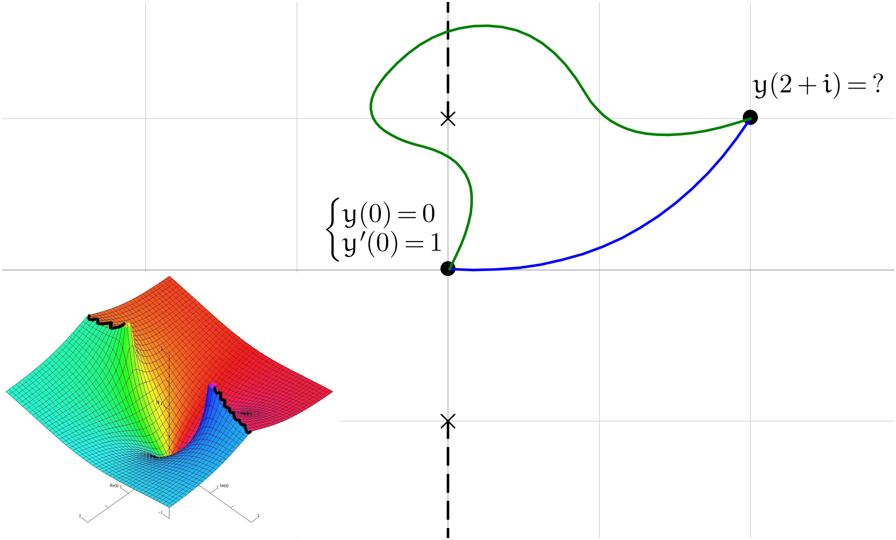

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```
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz

sage: dop.numerical_solution(ini=[0,1], path=[0,2+i])

sage: CBF(2+i).arctan()

sage: dop.numerical_solution(
        ini=[0,1],
        path=[0,i-1,2*i,2+i])
```

Analytic Continuation

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad [y(z)=\arctan(z)]$$

```
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz

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[1.1780972450961725 +/- 3.76e-17]
                    + [0.17328679513998633 +/- 3.81e-18]*I

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sage: CBF(2+i).arctan()

[1.178097245096172 +/- 5.86e-16]
                    + [0.1732867951399863 +/- 2.85e-17]*I

sage: dop.numerical_solution(
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        path=[0,i-1,2*i,2+i])
```

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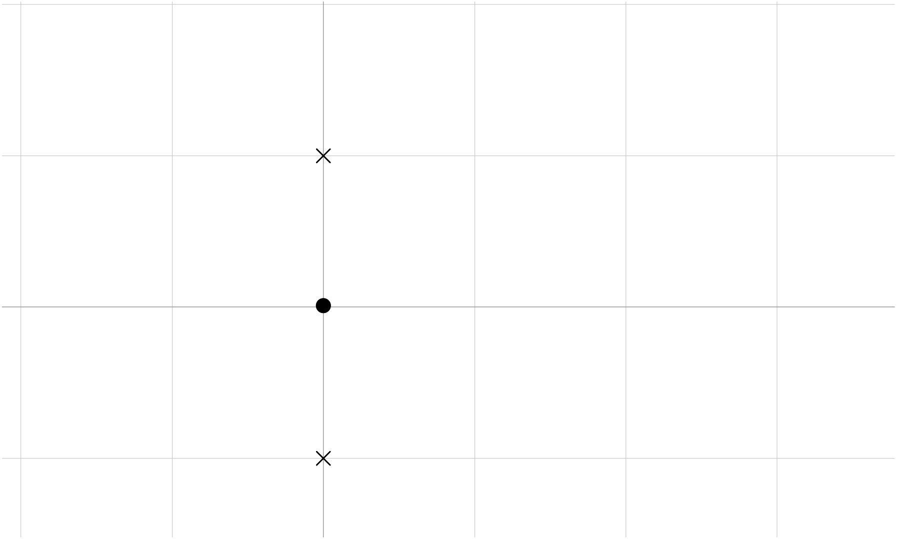
[1.178097245096172 +/- 5.86e-16]
                    + [0.173286795139986 +/- 2.85e-17]*I

sage: dop.numerical_solution(
        ini=[0,1],
        path=[0,i-1,2*i,2+i])

[-1.9634954084936208 +/- 3.03e-17]
                    + [0.1732867951399863 +/- 6.59e-18]*I
```

Transition Matrices

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0$$



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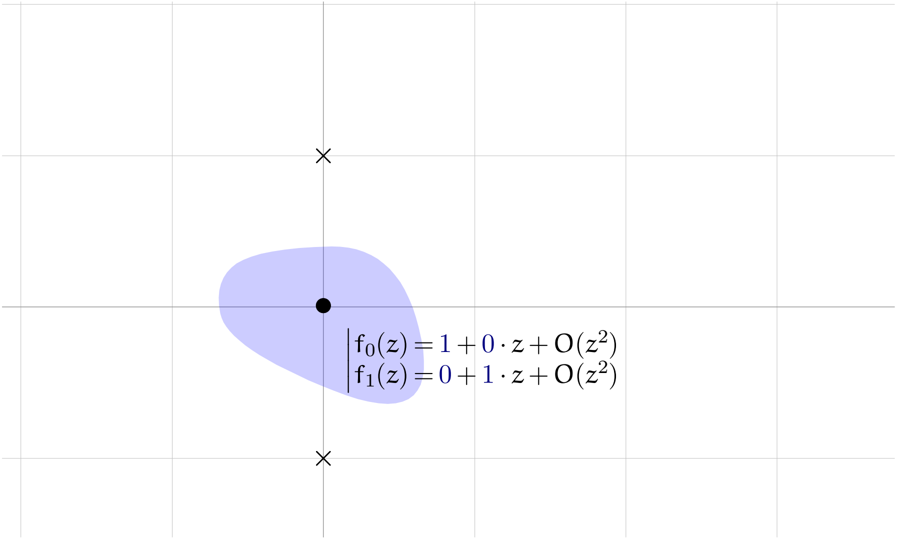
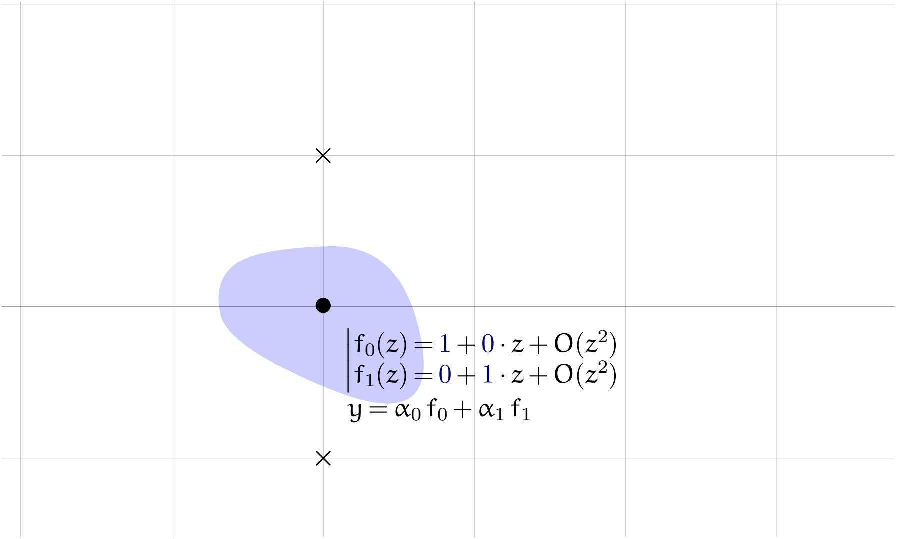


Transition Matrices

$(z^2+1)\,y''(z) + 2\,z\,y'(z) = 0$



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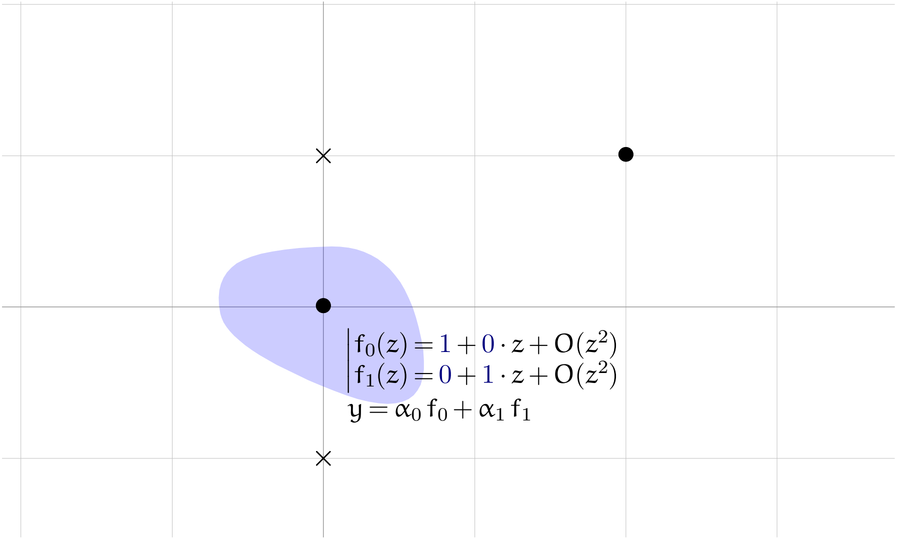

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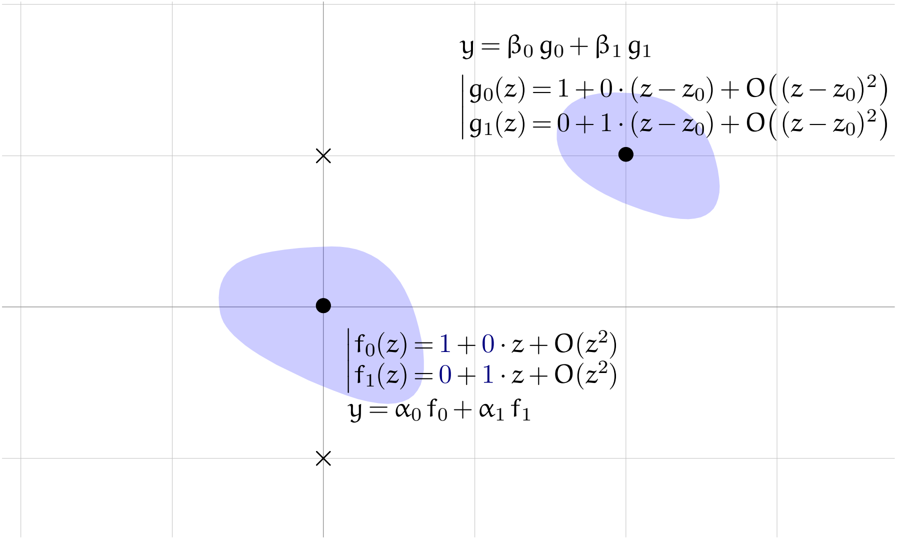

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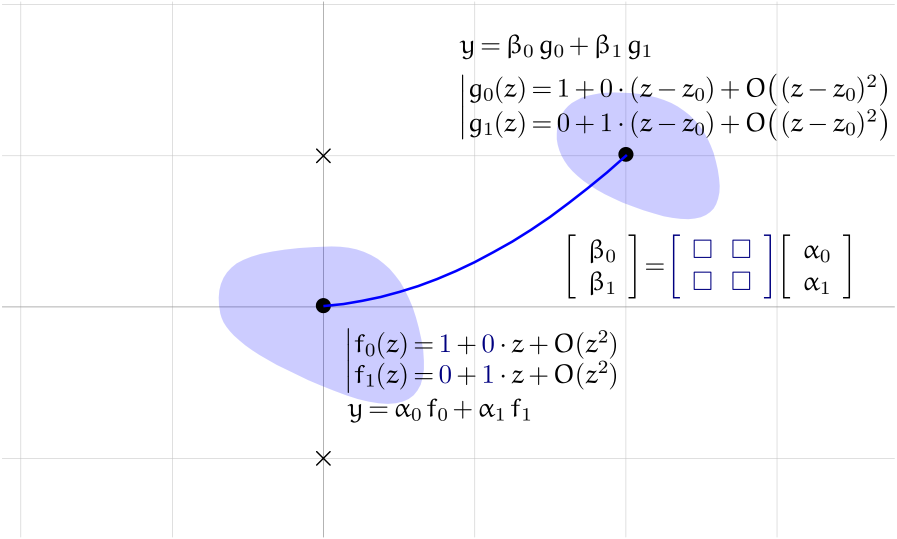

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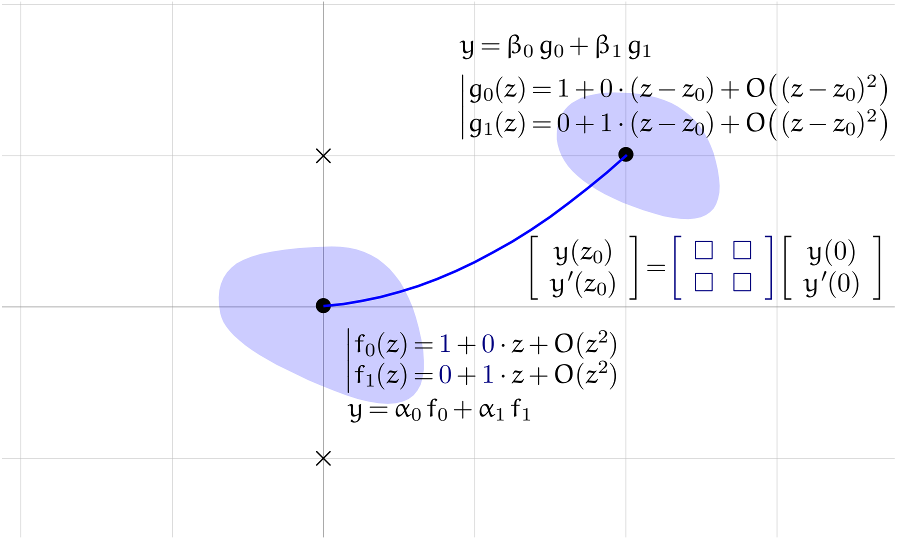

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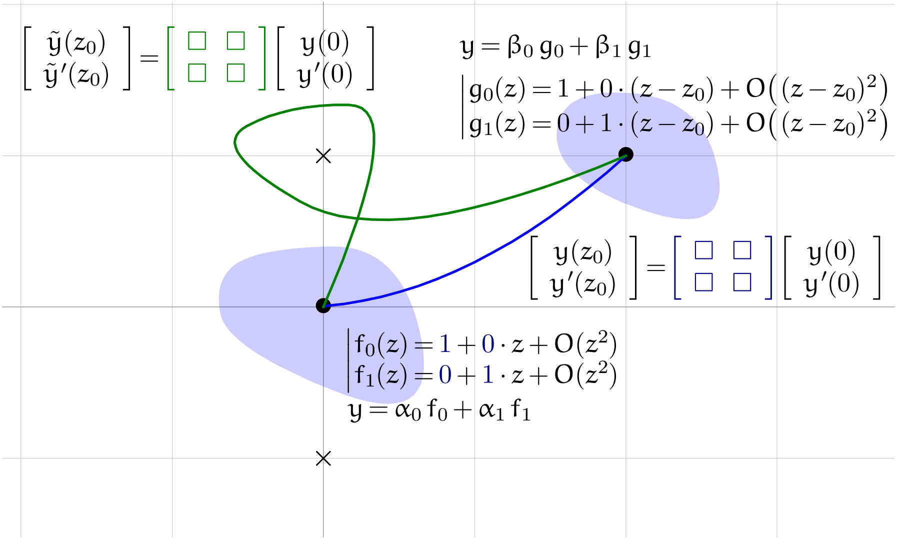


Transition Matrices

$$(z^2+1)\,y''(z) + 2\,z\,y'(z) = 0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

↗ "transition matrix"

```
sage: dop.numerical_transition_matrix([0,1])
sage: n(pi/4)
```

$$f_0(z) = 1 \qquad = 1 + 0 \cdot z + O(z^2)$$
$$f_1(z) = \arctan(z) \qquad = 0 + 1 \cdot z + O(z^2)$$

$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f_0(1) & f_1(1) \\ f_0'(1) & f_1'(1) \end{bmatrix}$$

Transition Matrices

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

↗ "transition matrix"

```
sage: dop.numerical_transition_matrix([0,1])
[   1.0000000000000000 [0.78539816339744831 +/- 4.94e-19]]
[                    0 [0.50000000000000000 +/- 1.44e-19]]
sage: n(pi/4)
```

$$f_0(z)=1 \qquad =1+0\cdot z+O(z^2)$$
$$f_1(z)=\arctan(z) \qquad =0+1\cdot z+O(z^2)$$

$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f_0(1) & f_1(1) \\ f_0'(1) & f_1'(1) \end{bmatrix}$$

Transition Matrices

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

↗ "transition matrix"

```
sage: dop.numerical_transition_matrix([0,1])
[   1.000000000000000 [0.78539816339744831 +/- 4.94e-19]]
[                   0 [0.50000000000000000 +/- 1.44e-19]]
sage: n(pi/4)
0.785398163397448
```

$$f_0(z)=1 \qquad =1+0\cdot z+O(z^2)$$
$$f_1(z)=\arctan(z) \qquad =0+1\cdot z+O(z^2)$$

$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f_0(1) & f_1(1) \\ f_0'(1) & f_1'(1) \end{bmatrix}$$

Regular Singular Points

The previous examples only involved **ordinary** (= non-singular) points.

$$z^2 y''(z) + z\,y'(z) + (z^2 - \nu^2)\,y(z) = 0 \qquad \text{(Bessel eq.)}$$

↖

singular point at 0
regular (≈ tame) in this case

✓ $z^{-3/2}\log z$ ✓ $z^{i\sqrt{2}}$ × $e^{\pm 1/z}$

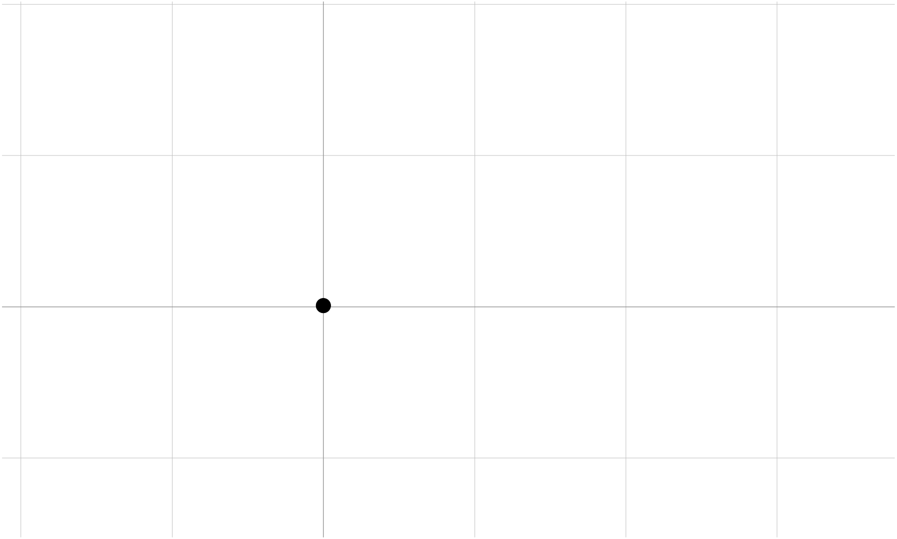
Theorem [Fuchs, 1866]

Assume that 0 is a regular singular point. Then, for some neighborhood D of 0, there exists a basis of solutions defined on $D\setminus\{0\}$ of the form

$$z^\lambda\,(y_0(z) + y_1(z)\log z + \cdots + y_t(z)\log^t z), \qquad \lambda \in \bar{\mathbb{Q}}, \qquad y_i \text{ analytic } \textbf{on } \mathbf{D}.$$

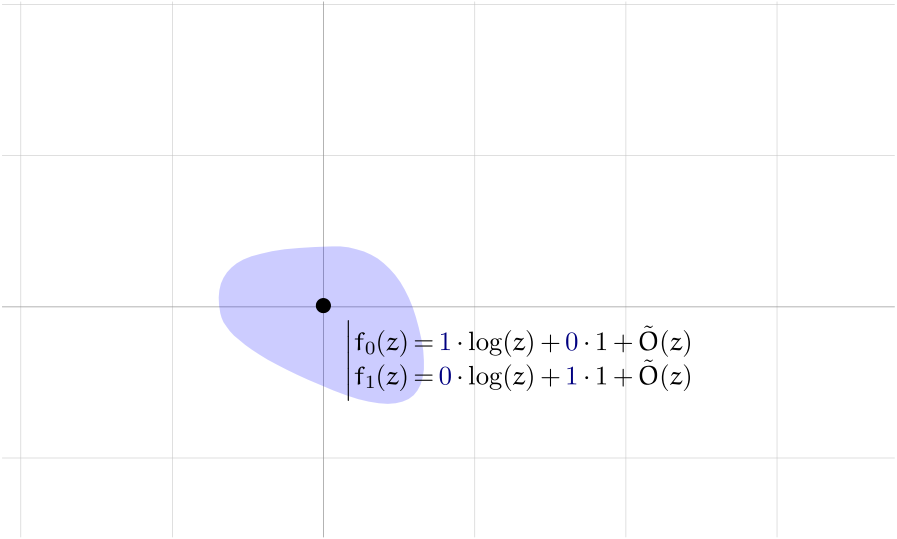
Regular Singular Connection Problems

$$z\,y''(z)+y'(z)+z\,y(z)=0 \quad [y(z)=J_0(z),Y_0(z)]$$



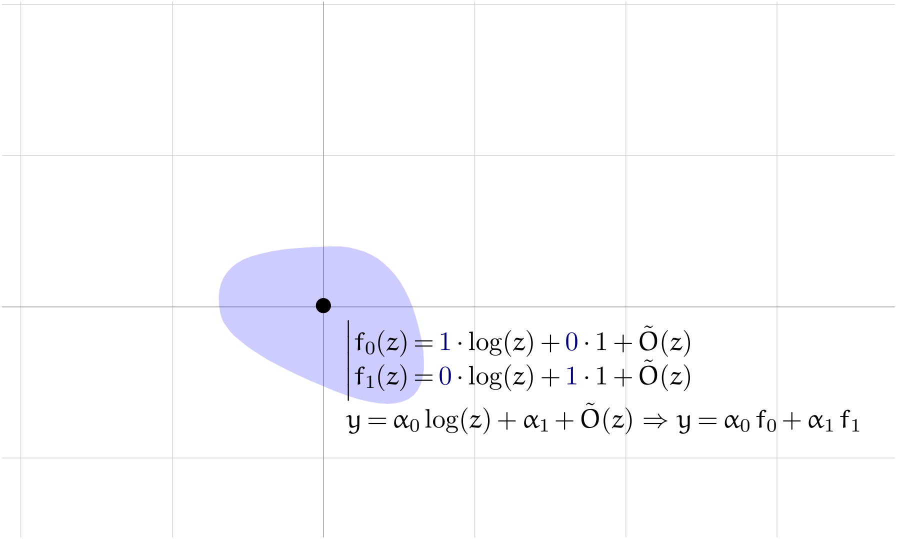
Regular Singular Connection Problems

$z\,y''(z) + y'(z) + z\,y(z) = 0 \quad [y(z) = J_0(z), Y_0(z)]$



Regular Singular Connection Problems

$z\,y''(z) + y'(z) + z\,y(z) = 0 \quad [y(z) = J_0(z), Y_0(z)]$

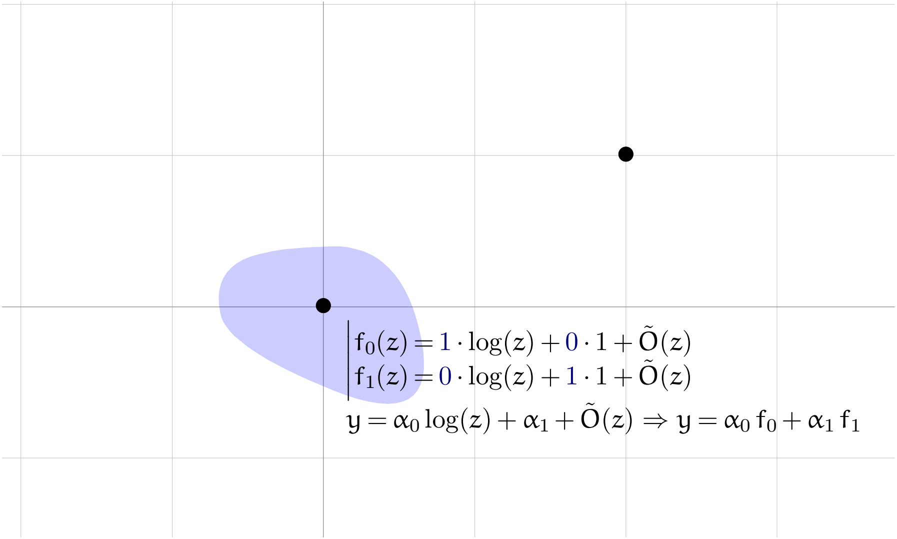


$$\begin{aligned} f_0(z) &= 1 \cdot \log(z) + 0 \cdot 1 + \tilde{O}(z) \\ f_1(z) &= 0 \cdot \log(z) + 1 \cdot 1 + \tilde{O}(z) \end{aligned}$$

$$y = \alpha_0 \log(z) + \alpha_1 + \tilde{O}(z) \Rightarrow y = \alpha_0\, f_0 + \alpha_1\, f_1$$

Regular Singular Connection Problems

$$z\,y''(z) + y'(z) + z\,y(z) = 0 \quad [y(z) = J_0(z), Y_0(z)]$$



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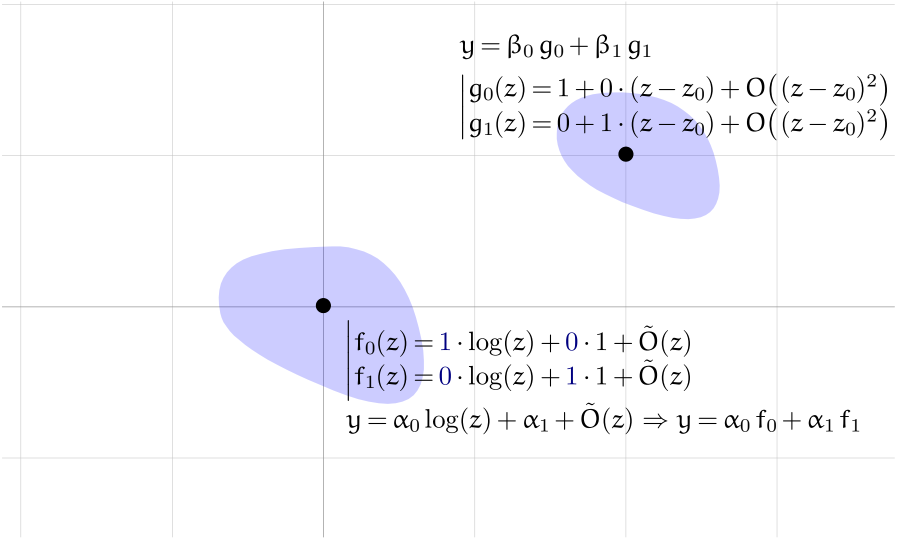

$$y = \beta_0\, g_0 + \beta_1\, g_1$$

$$\left| \begin{array}{l} g_0(z) = 1 + 0 \cdot (z - z_0) + O\big((z-z_0)^2\big) \\ g_1(z) = 0 + 1 \cdot (z - z_0) + O\big((z-z_0)^2\big) \end{array} \right.$$

$$\left| \begin{array}{l} f_0(z) = 1 \cdot \log(z) + 0 \cdot 1 + \tilde{O}(z) \\ f_1(z) = 0 \cdot \log(z) + 1 \cdot 1 + \tilde{O}(z) \end{array} \right.$$

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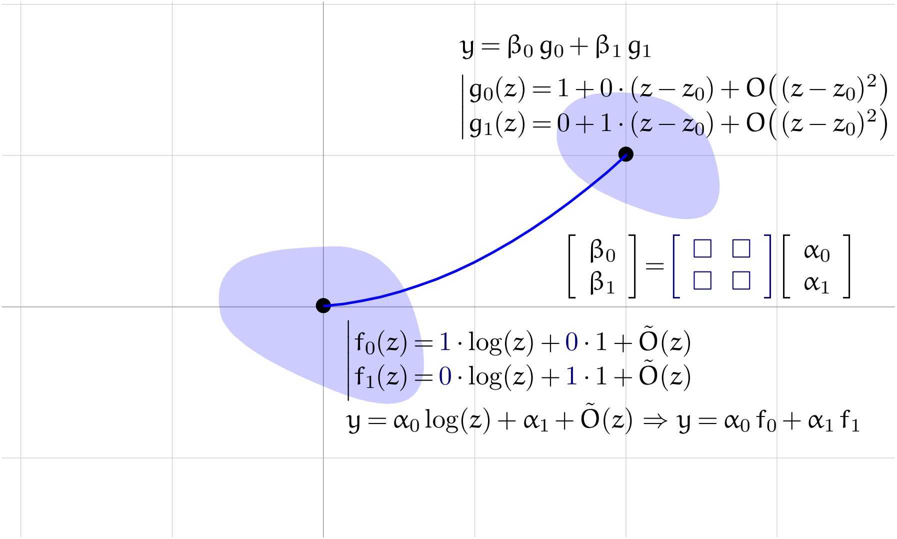

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$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$\left| \begin{array}{l} f_0(z) = 1 \cdot \log(z) + 0 \cdot 1 + \tilde{O}(z) \\ f_1(z) = 0 \cdot \log(z) + 1 \cdot 1 + \tilde{O}(z) \end{array} \right.$$

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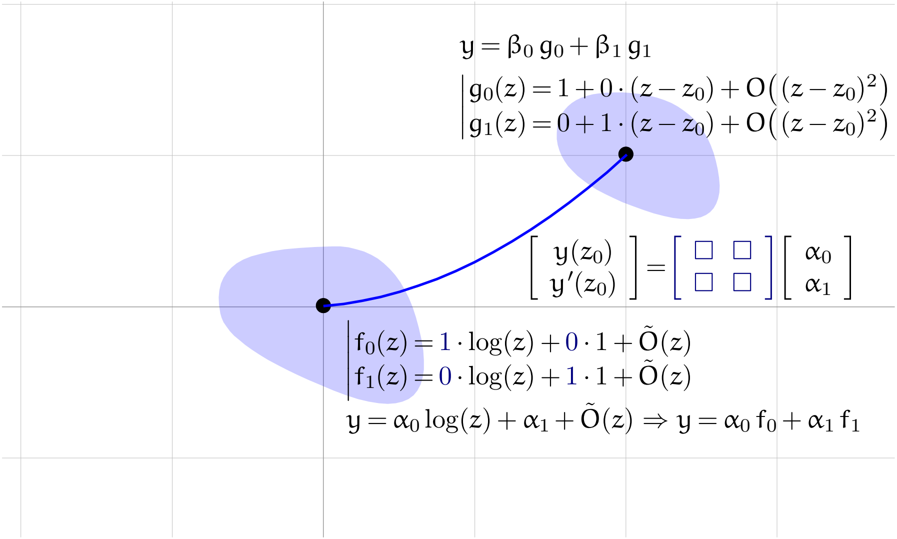

Regular Singular Connection Problems

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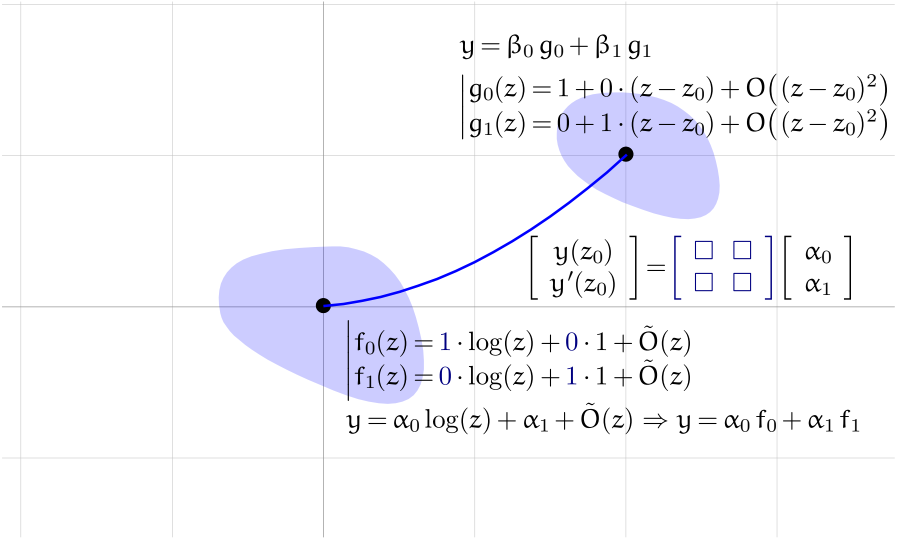

Regular Singular Connection Problems

$$z\,y''(z) + y'(z) + z\,y(z) = 0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where $y(z) = a \cdot \log z + b \cdot 1 + O(z)$

```
sage: dop = z*Dz^2 + Dz + z

sage: dop.local_basis_expansions(0)

sage: dop.numerical_transition_matrix([0, 1], 1e-10)
```

Applications: special functions, analytic combinatorics, resummation...

Regular Singular Connection Problems

$$z\,y''(z)+y'(z)+z\,y(z)=0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

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```
sage: dop = z*Dz^2 + Dz + z

sage: dop.local_basis_expansions(0)

 [log(z) - 1/4*z^2*log(z) + 1/4*z^2 + 1/64*z^4*log(z) - 3/128*z^4,
  1 - 1/4*z^2 + 1/64*z^4]

sage: dop.numerical_transition_matrix([0, 1], 1e-10)
```

Applications: special functions, analytic combinatorics, resummation...

Regular Singular Connection Problems

$$z\,y''(z)+y'(z)+z\,y(z)=0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

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sage: dop.numerical_transition_matrix([0, 1], 1e-10)

[ [0.22734424279 +/- 4.98e-12]  [0.76519768656 +/- 2.04e-12]]
[  [1.1761104988 +/- 2.83e-11] [-0.44005058574 +/- 4.94e-12]]
```

Applications: special functions, analytic combinatorics, resummation...

Face-Centered Cubic Lattices

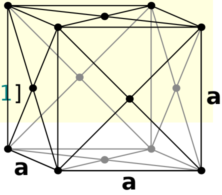
[Koutschan 2013]

```
sage: dop4 = ((-1 + z)*z^3*(2 + z)*(3 + z)*(6 + z)*(8 + z)*(4 +
      3*z)^2*Dz^4 + 2*z^2*(4 + 3*z)*(-3456 - 2304*z + 3676*z^2 +
      4920*z^3 + 2079*z^4 + 356*z^5 + 21*z^6)*Dz^3 + 6*z*(-5376
      - 5248*z + 11080*z^2 + 25286*z^3 + 19898*z^4 + 7432*z^5
      + 1286*z^6 + 81*z^7)*Dz^2 + 12*(-384 + 224*z + 3716*z^2
      + 7633*z^3 + 6734*z^4 + 2939*z^5 + 604*z^6 + 45*z^7)*Dz +
      12*z*(256 + 632*z + 702*z^2 + 382*z^3 + 98*z^4 + 9*z^5))

sage: dop4.local_basis_monomials(0)

sage: dop4.local_basis_monomials(1)

sage: dop4.numerical_transition_matrix([0,1])[0,-1]
```



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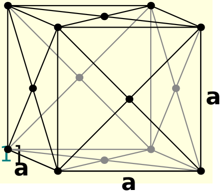
sage: dop4.local_basis_monomials(0)

 [1/6*log(z)^3, 1/2*log(z)^2, log(z), 1]

sage: dop4.local_basis_monomials(1)

sage: dop4.numerical_transition_matrix([0,1])[0,-1]
```



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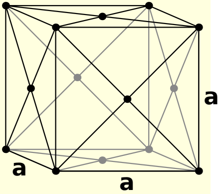
      12*z*(256 + 632*z + 702*z^2 + 382*z^3 + 98*z^4 + 9*z^5))

sage: dop4.local_basis_monomials(0)
 [1/6*log(z)^3, 1/2*log(z)^2, log(z), 1]

sage: dop4.local_basis_monomials(1)
 [1, (z - 1)*log(z - 1), z - 1, (z - 1)^2]

sage: dop4.numerical_transition_matrix([0,1])[0,-1]
```



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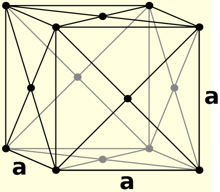
sage: dop4.numerical_transition_matrix([0,1])[0,-1]
[1.1058437979212048 +/- 3.99e-17] + [+/- 6.96e-26]*I
```



To Do: Features

Core features

- ✓ Analytic continuation
- ✓ Regular singular points
- Irregular singular points

Evaluation points

- ✓ Rationals, $\mathbb{Q}[i]$
- ✓ Algebraic numbers
- ✓ Small intervals
- Wide intervals
- Series ?

Initial values

- ✓ Numeric, interval
- ✓ Exact

User interface

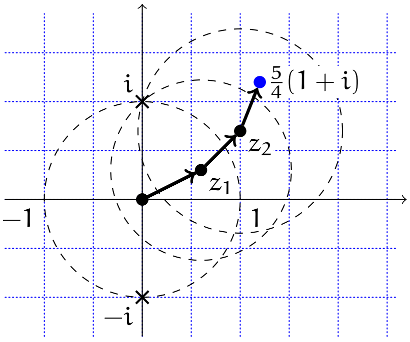
- ✓ Evaluation
- ✓ Transition matrices
- Branch cuts
- Function objects

Applications

- ✓ Polynomial approximations
- Singularity analysis
- Zeros
- ...

3 *How* it does it

A Taylor Series Method



$$\arctan\left(\frac{5}{4}(1+i)\right)?$$

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\ldots + 0.22\ldots \\ 0 & 0.72\ldots - 0.20\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.39\ldots + 0.24\ldots \\ 0 & 0.57\ldots - 0.29\ldots \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$

...

- Locally, the solutions are given by **convergent power series**
- **Sum the series** numerically to get “initial values” at a new point

- Large steps ($\propto$ radius of convergence)
- Extends to the regular singular case

Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$
obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n)\, y_{n+s} + \cdots + b_1(n)\, y_{n+1} + b_0(n)\, y_n = 0.$$

(And conversely, for D-finite formal power series.)

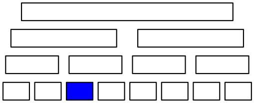
- Easy to generate
- Also leads to **fast algorithms**

[Schroeppel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988;
van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

time $\mathbf{O}(\mathbf{M}(n \log^2 n))$, space $\mathbf{O}(n)$

for fixed z and $\varepsilon = 2^{-n}$



Recurrences

The **coefficients** of a D-finite function $\sum_{\nu \in \lambda + \mathbb{Z}} \sum_{k=0}^{K} y_{\nu,k} z^{\nu} \frac{\log(z)^k}{k!}$
obey a linear **recurrence relation** with polynomial coefficients:

$$[b_s(\nu + S_k) \cdot S_\nu^s + \cdots + b_1(\nu + S_k)\, S_\nu + b_0(\nu + S_k)] \cdot (y_{\nu,k}) = 0.$$

- Easy to generate
- Also leads to **fast algorithms**

[Schroeppel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988;
van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

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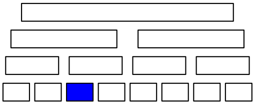

Error Bounds

Round-off errors

Real & complex arithmetic based on **Arb** [Johansson 2012–]

(`{Real,Complex}BallField` in Sage)

- More generally: takes care of error propagation
- Arb supports truncated power series (cf. autodiff)
- Manual error analysis still useful when intervals blow up

Truncation Errors

$$\sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\text{known}} + \underbrace{\sum_{n=N}^{\infty} u_n z^n}_{|\cdot| \leqslant ?}$$

- Majorant series
- ”Adaptive” bounds using residuals

Majorant Series

[Cauchy 1842; ...; van der Hoeven 2001; M. & Salvy 2010]

- Instead of directly bounding $|\sum_{n \geqslant N} u_n z^n|$, compute a **majorant series**:

$$\sum v_n z^n \in \mathbb{R}_{\geqslant 0}[[z]] \qquad \text{s.t.} \qquad \forall n, \quad |u_n| \leqslant v_n$$

- Bound the differential equation with a simple "**model equation**":

$$L(z, D_z) \cdot u = 0 \qquad \lll \qquad v'(z) - \frac{1}{(1 - \alpha z)} v(z) = 0$$

for us: always 1st order

- Solve the model equation and study the solutions:

$$v(z) = \exp \int^z \frac{dt}{1 - \alpha t} \qquad \left| \sum_{n=N}^{+\infty} u_n z^n \right| \leqslant \sum_{n=N}^{+\infty} v_n |z|^n \leqslant ?$$

"Adaptive" Bounds

Analogy: $$A\,x = b \qquad \|A^{-1}\| \leqslant M$$ $$A \in \mathrm{GL}_n(\mathbb{C})$$

$$A\,\tilde{x} = \tilde{b} \qquad \Rightarrow \qquad \|x - \tilde{x}\| \leqslant M \cdot \underbrace{\|b - \tilde{b}\|}_{\text{known}}$$

Idea: do something similar when A is a **differential operator**

Adaptive Majorants

$L(z, D_z) \cdot u = 0$

Residual: $q(z) := L(z, D_z) \cdot \tilde{u}$

$$u(z) = \sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\tilde{u}(z)} + \sum_{n=N}^{\infty} u_n z^n$$

- **Model equation**

$$\begin{array}{rcl} q(z) & \lll & \hat{q}(z) \\ L(z, D_z) \cdot (\tilde{u} - u) = q & \lll & \hat{L}(z, D_z) \cdot v = \hat{q} \end{array}$$

- **Majorant property:**

$$(\forall n \leqslant n_0) \quad |u_n| \leqslant v_n \qquad \Rightarrow \qquad (\forall n) \quad |u_n| \leqslant v_n$$

- **Solving the model equation**

$$v(z) = h(z) \left(\text{cst} + \int^z \frac{t^{-1} \hat{q}(t)}{h(t)} \, dt \right) \qquad \text{where } h(z) = \exp \int^z t^{-1} \hat{a}(t) \, dt$$

choose $\text{cst} = 0$; $= O(z^N)$

To Do: Efficiency

- **Series summation algorithms**
  - Floating-point summation with manual error analysis
  - Rectangular splitting
  - (Truncated) binary splitting @ reg. sing.
  - Bit-burst method
- **Analytic continuation algorithms**
  - Automatic path optimization
  - Analytic continuation of individual solutions
  - Simultaneous computation of local solutions @ reg. sing.
- **Implementation**
  - Compiled version of direct summation
  - Optimizations
  - Faster basic arithmetic (in Sage)

Summary

What it is: an extension of ore_algebra written in/for SageMath

What it does: numerical analytic continuation & singular connection, for arbitrary D-finite functions, with rigorous error bounds

How it works: Taylor series, analytic continuation, recurrences, adaptive majorants, ball arithmetic

Code available at

`http://kauers.de/software.html`

`http://marc.mezzarobba.net/code/ore_algebra-analytic`

Perspectives

Faster algorithms, lower-level code,

D-finite functions as objects,

irregular singular connection problems...

Bug reports, feature requests, examples welcome!

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Asymptotics of Apéry Numbers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad b_n = \sum_{k=1}^{n} \left(\frac{a_n}{k^3} - \sum_{m=1}^{k} \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2\, m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

(1, 5, 73, 1445, 33001...) (0, 6, 351/4, 62531/36, ...)

- The OGS $a(z)$ and $b(z)$ are solutions of $L = z^2\,(z^2 - 34\,z + 1)\,D_x^4 + \cdots$
- Singular points: 0, $\alpha = \left(\sqrt{2}+1\right)^4 \approx 33.9$, $\alpha^{-1} = \left(\sqrt{2}-1\right)^4 \approx 0.0294$
- Prove: $a_n, b_n = \alpha^{n+o(n)}$ $\qquad b_n - \zeta(3)\,a_n = \alpha^{-n+o(n)}$
- Local expansion at α^{-1}: $a(z) = c_0\, f_0(z) + c_1\, f_1(z) + c_2\, f_2(z) + c_3\, f_3(z)$

  where $f_0(\alpha^{-1}+t) = 1 + O(t^3)$ $\qquad f_3(\alpha^{-1}+t) = t + O(t^3)$

  $f_1(\alpha^{-1}+t) = \sqrt{t} + O(t^3)$ $\qquad f_4(\alpha^{-1}+t) = t^2 + O(t^3)$
- Singularity analysis: $(c_1 \neq 0)$

$$a(z) \sim c_1 \sqrt{z - \alpha^{-1}} \qquad \Rightarrow \qquad a_n \sim c_1\,[z^n]\sqrt{z-\alpha^{-1}} \sim \frac{c_1\, i}{2\sqrt{\alpha\,\pi}}\,\alpha^n\, n^{-3/2}$$

Recurrences and Regular Singular Points

$$y(z) = \sum_{n=0}^{\infty} y_n z^n$$

$$L(z, z\,D_z) \cdot y = 0 \quad \Leftrightarrow \quad L(S_n^{-1}, n) \cdot (y_n)_{n \in \mathbb{Z}} = 0$$

$$y(z) = \sum_{n \in \lambda + \mathbb{Z}} \sum_{k=0}^{K} y_{n,k}\, z^n \frac{\log(z)^k}{k!}$$

$$L(z, z\,D_z) \cdot y = 0 \quad \Leftrightarrow \quad L(S_n^{-1}, n + S_k) \cdot (y_{n,k})_{n \in \mathbb{Z}, k \in \mathbb{N}} = 0$$

Usage

- Binary splitting
- Bound computations
- ...