Numerical Evaluation of D-Finite Functions in SageMath

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A Non-Scientific Aside

Rigorous Multiple-Precision Evaluation of D-Finite Functions in SageMath

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Abstract. We present a new open source implementation in the SageMath computer algebra system of algorithms for the numerical solution of linear ODEs with polynomial coefficients. Our code supports regular singular connection problems and provides rigorous error bounds.

1 Introduction

Many special functions satisfy differential equations

$$p_r(x)f^{(r)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0 \qquad (1)$$

whose coefficients p_i depend polynomially on the variable x. In virtually all cases, such special functions can be defined by complementing (1) either with simple initial values $f(0), f'(0), \ldots$ or with constraints on the asymptotic behavior of $f(x)$ as x approaches a singular point. For example, the error function satisfies

$$\operatorname{erf}''(x) + 2x\operatorname{erf}'(x) = 0, \qquad \operatorname{erf}(0) = 0, \quad \operatorname{erf}'(0) = \frac{2}{\sqrt{\pi}}, \qquad (2)$$

while the modified Bessel function K_0 is the solution of

$$xK_0''(x) + K_0'(x) - xy(x) = 0 \quad \text{s.t.} \quad K_0(x) = -\log(x/2) - \gamma + \mathrm{O}_{x\to 0}(x). \quad (3)$$

This observation has led to the idea of developing algorithms that deal with these functions in a uniform way, using the ODE (1) as a data structure [11,2].

In this context, solutions of linear ODEs with polynomial coefficients are called *D-finite* (or *holonomic*) functions. These names originate from combinatorics, where D-finite power series arise naturally as generating functions [17,4]. While classical special functions typically satisfy ODEs of order 2 to 4 with simple

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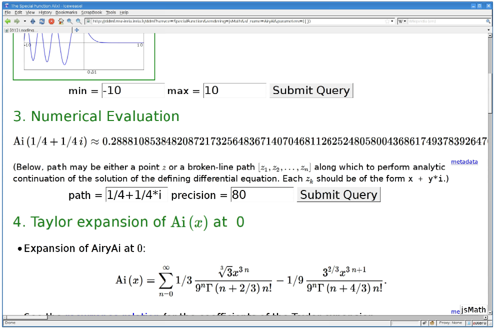
What it Is: A Successor for NumGfun



$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\, \frac{\sqrt[3]{3}x^{3\,n}}{9^n \Gamma\,(n+2/3)\,n!} - 1/9\, \frac{3^{2/3} x^{3\,n+1}}{9^n \Gamma\,(n+4/3)\,n!}.$$

http://ddmf.msr-inria.inria.fr

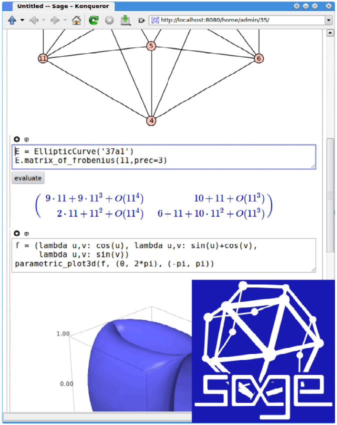
[Benoit, Chyzak, Darrasse, Gerhold, Grégoire, Koutschan, M., Salvy 2010–]

NumGfun [M. 2010]

- General D-Finite functions
- Arbitrary precision
- Rigorous error bounds

- Maple
- Oriented towards special functions

What it Is: A SageMath Implementation



- Python library
- "A viable alternative to Magma, Maple, Mathematica and Matlab"

```
sage: Pols.<z> = PolynomialRing(QQ)
sage: Pols
    Univariate Polynomial Ring in z
    over Rational Field
sage: (z + 1)*(z-1)
    z^2 - 1
```

http://sagemath.org/

GNU GPL v2+

What it Is: Based on ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

```
sage: from ore_algebra import OreAlgebra
sage: DiffOps.<Dz> = OreAlgebra(Pols)
sage: DiffOps
    Univariate Ore algebra in Dz over Univariate Polynomial Ring
    in z over Rational Field
sage: Dz*z
    z*Dz + 1
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

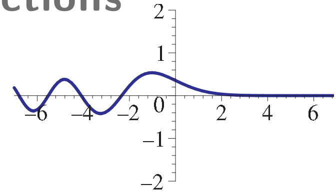
http://www.risc.jku.at/research/combinat/software/ore_algebra/

What it Is: The -analytic Branch

- **Symbolic-numeric** extensions for ore_algebra
- Real & complex arithmetic based on **Arb** [Johansson 2012–]
  (`{Real,Complex}BallField` in Sage)
- Both for "end users" and for prototyping algorithms
- Development branch, not (yet) integrated into any release of ore_algebra

http://marc.mezzarobba.net/code/ore_algebra-analytic
GNU GPL v2+

What it Does: Special Functions




```
sage: diffop = Dz^2 - z
sage: diffop.numerical_solution(
          [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
          [0, i], 1e-40)
  [0.3314933054321411889845293326171343458866 +/- 5.51e-41] +
  [-0.31744985896844377347764292790925852645896 +/- 7.23e-42]*I
sage: ComplexBallField(138)(i).airy_ai()
  [0.33149330543214118898452933261713434588655 +/- 5.25e-42] +
  [-0.31744985896844377347764292790925852645896 +/- 1.59e-42]*I
```

What it Does: Polynomial Approximations

```
sage: diffop
```

```
Dz^2 - z
```

```
sage: from ore_algebra.analytic import
      polynomial_approximation as polapprox
sage: polapprox.on_interval(diffop,
        [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
        [[-1,1]], 1e-10)
```

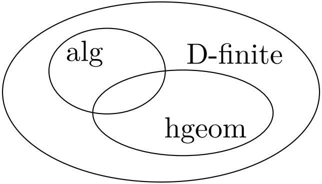
```
[-0.0005136124318 +/- 3.11e-14]*z^7 + [0.001972375574 +/-
5.11e-13]*z^6 + [2.373692757e-8 +/- 1.52e-18]*z^5 + [-
0.021568281826 +/- 7.76e-13]*z^4 + [0.05917133936 +/- 2.27e-
12]*z^3 + [-3.780865636e-10 +/- 3.16e-20]*z^2 + [-
0.2588194037 +/- 2.12e-11]*z + [0.3550280539 +/- 2.94e-11]
```

What it Does: D-Finite Functions

[Stanley, Zeilberger... 1980–]

An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

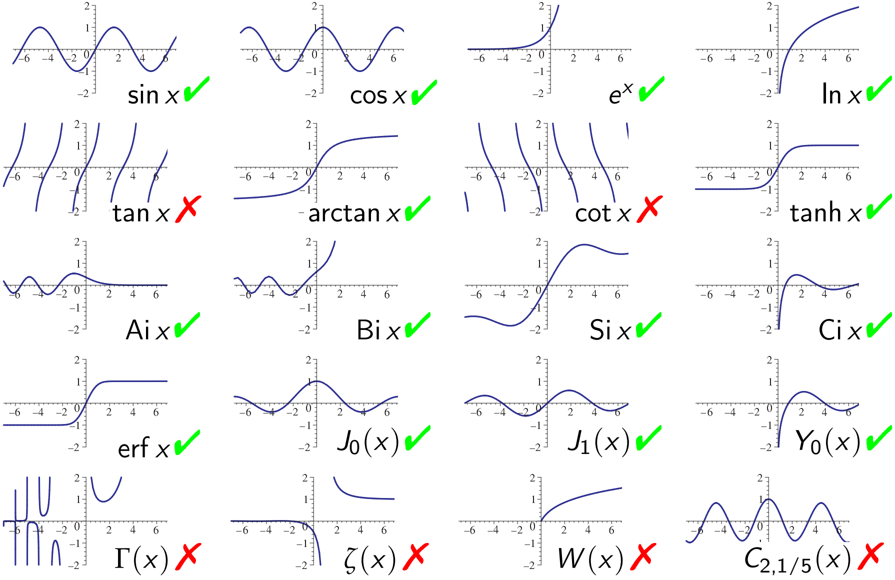
$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z)\, y(z) = 0, \qquad a_j \in \mathbb{K}[z]$$



Philosophy:

Provide **general algorithms** for D-finite functions, using { ODE + initial values } as a data structure.

What it Does: D-Finite Functions



What it Does: D-Finite Functions

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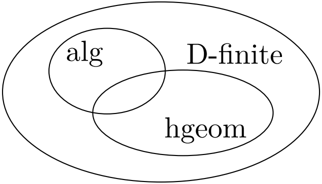

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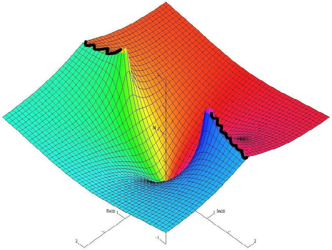
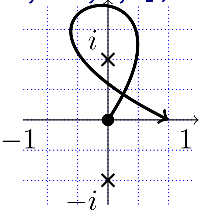
What is Does: Analytic Continuation



```
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz

sage: dop.numerical_solution(
        ini=[0,1], path=[0,1])

  [0.78539816339744831 +/- 1.08e-18]

sage: dop.numerical_solution(
        ini=[0,1],
        path=[0,i+1,2*i,i-1,0,1])

  [3.9269908169872415
        +/- 4.81e-17]
  + [+/- 4.63e-21]*I
```



$y(z) = \arctan(z)$

$(z^2+1)\, y''(z) + 2\, z\, y'(z) = 0$

What it Does: Transition Matrices

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

```
sage: dop.numerical_transition_matrix([0,1])
   [    1.0000000000000000 [0.78539816339744831 +/- 3.85e-19]]
   [                     0    [0.5000000000000000 +/- 3e-22]]
```

$$\begin{aligned} f(z) &= 1 & &= 1 + 0\cdot z + O(z^2) \\ g(z) &= \arctan(z) & &= 0 + 1\cdot z + O(z^2) \end{aligned} \qquad \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f(1) & g(1) \\ f'(1) & g'(1) \end{bmatrix}$$

What it Does: Regular Singular Points

The previous examples only involved **ordinary** (= non-singular) points.

$$z^2\, y''(z) + z\, y'(z) + (z^2 - \nu^2)\, y(z) = 0$$

0 **singular** point; **regular** in this case

Theorem [Fuchs, 1866]

Assume that 0 is a regular singular point. Then, for some $D \ni 0$, there exists a basis of solutions defined on $D \backslash \{0\}$ of the form

$z^{\lambda}\,(y_0(z) + y_1(z)\log z + \cdots + y_t(z)\log^t z), \qquad \lambda \in \bar{\mathbb{Q}}, \qquad y_i$ **analytic** on D.

Examples: $z^{\sqrt{2}}$, $z^{-3/2}\log z$

What it Does: Regular Singular Connection Problems

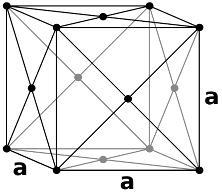
```
sage: dop = z*Dz^2 + Dz + z
sage: dop.local_basis_monomials(0)
    [log(z), 1]
sage: dop.numerical_transition_matrix([0, 1], 1e-10)
   [ [0.22734424279 +/- 4.98e-12]  [0.76519768656 +/- 2.04e-12]]
   [  [1.1761104988 +/- 2.83e-11] [-0.44005058574 +/- 4.94e-12]]
```

$$\begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad \text{where } y(z) = a \log z + b + O(z)$$

Applications: special functions, analytic combinatorics, resummation...

Face-Centered Cubic Lattices [Koutschan 2013]

```
sage: dop4 = ((-1 + z)*z^3*(2 + z)*(3 + z)*(6 + z)*(8 +
      z)*(4 + 3*z)^2*Dz^4 + 2*z^2*(4 + 3*z)*(-3456 - 2304*z +
      3676*z^2 + 4920*z^3 + 2079*z^4 + 356*z^5 + 21*z^6)*Dz^3
      + 6*z*(-5376 - 5248*z + 11080*z^2 + 25286*z^3 +
      19898*z^4 + 7432*z^5 + 1286*z^6 + 81*z^7)*Dz^2 + 12*(-
      384 + 224*z + 3716*z^2 + 7633*z^3 + 6734*z^4 + 2939*z^5
      + 604*z^6 + 45*z^7)*Dz + 12*z*(256 + 632*z + 702*z^2 +
      382*z^3 + 98*z^4 + 9*z^5))

sage: dop4.local_basis_monomials(0)

  [1/6*log(z)^3, 1/2*log(z)^2, log(z), 1]

sage: dop4.local_basis_monomials(1)

  [1, (z - 1)*log(z - 1), z - 1, (z - 1)^2]

sage: dop4.numerical_transition_matrix([0,1])[0,-1]

  [1.1058437979212048 +/- 3.99e-17] + [+/- 6.96e-26]*I
```



Face-Centered Cubic Lattices [Koutschan 2013]

dop6 $= 410085196915322880\,z^{35} + 1129052664742115635 20\,z^{34} + 1171669263761496\backslash$
$1489920\,z^{33} + 69081740128707891736 3200\,z^{32} + 272048626438466115227 61600\,z^{31} +$
$778811406918247228618497600\,z^{30} + 1704438412411524078142979 2800\,z^{29} + 294245234\backslash$
$066850000428339092800\,z^{28} + 4083424587805117060272476125800\,z^{27} + 459730295491197962\backslash$
$35386142827300\,z^{26} + 4196955988908982532034558767499 30\,z^{25} + 3064297176174091671717 9\backslash$
$85958725620\,z^{24} + 17169584489259696388755804636033570\,z^{23} + 645817719616848100772794\backslash$
$75394020500\,z^{22} + 517142219342720994204761262167667 00\,z^{21} - 1473967391504437899277 38\backslash$
$0487903179960\,z^{20} - 14237554341321335335392023192872385940\,z^{19} - 832163401343931150 1\backslash$
$683422098038445434 0\,z^{18} - 3640191543281075625688479068224880635 50\,z^{17} - 126157147851\backslash$
$340108817703509327552630430 0\,z^{16} - 352834103209889699532343901711795685615 0\,z^{15} -$
$796436951859377802952105607044279446690 0\,z^{14} - 142805007261627862547128411638750 0\backslash$
$1728600\,z^{13} - 1953465311568634254358083196094197891800 0\,z^{12} - 1839878333422238008423\backslash$
$801242870473196000 0\,z^{11} - 7553741785990309357234054786177488000000\,z^{10} + 88874323094\backslash$
$1952240398397617177569760000 0\,z^{9} + 21137039158366320685856256980012112000000\,z^{8} +$
$2268269355393480469044664729550880000000 0\,z^{7} + 149381834281462611905463546716160\backslash$
$00000000\,z^{6} + 46902465285848163299401994004480000000 00\,z^{5} - 872829008634785738926 16\backslash$
$2452480000000000\,z^{4} - 1104327940779745890150773145600000000000\,z^{3} - 353898708207580\backslash$
$856772919296000000000000 0\,z^{2} - 5207934291077444487413760000000000000 00\,z - 2428790621\backslash$
$9318850355200000000000000000 +$

Face-Centered Cubic Lattices [Koutschan 2013]

(3964156903514787840 z^{36} + 110471848996341353472 0 z^{35} + 1178710887399303528345 60 z^{34} +
7183287516644479615795200 z^{33} + 29310583521894290378185 5360 z^{32} + 870657237873\
4984776799502400 z^{31} + 197949776138115866133849254880 z^{30} + 355549462414631854804645\
3851120 z^{29} + 514578986720138650981112912473 20 z^{28} + 606522834979531840684521625 23\
7020 z^{27} + 583536683684602718287692085634 8950 z^{26} + 454552025018263589746062199 8197\
4015 z^{25} + 279153404467502062948531557838260750 z^{24} + 1252275399372837134061507042 62\
8908795 z^{23} + 294318280292355203816130758470 6940070 z^{22} − 104835131152062893985104 13\
216920199750 z^{21} − 169948182933507479161257565568616530700 z^{20} − 115496959477627 7160\
6490777859838205538 70 z^{19} − 5548694490781020038019823355124585193590 z^{18} −
20745229517577451272377158241970915439245 z^{17} − 622329639287946386594230696517 61\
724690290 z^{16} − 15081004590197893286416349304640546126 2105 z^{15} − 29252862652300566 16\
2939023688304685997 6150 z^{14} − 4413950960631831488390081722485803 37780300 z^{13} − 48412\
357876453704303186120647371526934 3000 z^{12} − 312976584649334763451810663858004196 42\
0000 z^{11} + 31415133499909950234831915395869293600000 z^{10} + 33073846008767455546849 14\
826295584680000 00 z^9 + 391096978918364972225128472061480072000000 z^8 + 232460170425\
948027345434850305279520000000 z^7 + 1806013493488429983484709934556800000000 0 z^6 −
100213400891192102370293326036992000000000 z^5 − 8385906451598513649590309945856 0\
000000000 z^4 − 29487051768804049058487328972800000000000 z^3 − 5516482719743042228 47\
284019200000000000 z^2 − 379553789663505987184558080000000000000 z + 83793276456650\
033725440000000000000000) Dz +

Face-Centered Cubic Lattices [Koutschan 2013]

$$
\begin{aligned}
&(813335640548723712 0\,z^{37} + 229413178204366431744 0\,z^{36} + 2512953285347621936332 80\,z^{35} + \\
&15795453015240816970091520\,z^{34} + 666093618246765502077439680\,z^{33} + 2046937571\backslash \\
&2416843040909376160\,z^{32} + 48181733826733863978332874912 0\,z^{31} + 8967973126212020032 51\backslash \\
&7991216960\,z^{30} + 13469691485430453672228186695430 0\,z^{29} + 1651833654984876079820125 88\backslash \\
&5678650\,z^{28} + 16607490026343429532811575311949230\,z^{27} + 136263869454304799146859253 3\backslash \\
&46813455\,z^{26} + 89586531932747144763811128987323871 0\,z^{25} + 448971107426438490652992 59\backslash \\
&90254793265\,z^{24} + 144918522834945778266540039325477116 90\,z^{23} - 16850906647119454698 3\backslash \\
&174648133542750\,z^{22} - 38626589454982688122912310447073109644 0\,z^{21} - 3163259131060568\backslash \\
&584546113343781987561220\,z^{20} - 16636182069413821170544684047556220568150\,z^{19} - \\
&66246740089393676080981537130378090658525\,z^{18} - 2090802458728506315663121374496195\backslash \\
&61543730\,z^{17} - 52909746574010477639177283467503394659333 5\,z^{16} - 10650386207573139293\backslash \\
&916393610323634537509 30\,z^{15} - 16536516446856201421670094221240225552217 00\,z^{14} - 1829\backslash \\
&383474513975929874027770563298831967800\,z^{13} - 108870989683690580666028414927762156\backslash \\
&8328000\,z^{12} + 43738406733732888694448396333695290408000 0\,z^{11} + 167838036536543200662\backslash \\
&54514732362690120000 00\,z^{10} + 15643853555920279359226831628986551120000 00\,z^{9} + 30360\backslash \\
&739871532595420703230366310784000000 0\,z^{8} - 9295542633845547711364838015847456000 0\backslash \\
&0000\,z^{7} - 133465853572648237153690804917964800000000 0\,z^{6} - 941977534006524837182879\backslash \\
&2635648000000000 00\,z^{5} - 35029765385278767758927501660160000000000 0\,z^{4} - 64286241473\backslash \\
&892148234640584704000000000000 00\,z^{3} - 3745850575154857509858508800000000000000\,z^{2} + \\
&6687067779833962474045440000000000000000 0\,z + 7286371865795655106560000000000000 0\backslash \\
&0000)\,\mathrm{D}z^{2} +
\end{aligned}
$$

Face-Centered Cubic Lattices [Koutschan 2013]

$(6219625486549063680\, z^{38} + 177553133630802252288\,0 z^{37} + 19940999663513258975232\,0 z^{36} +$

$1290486249759244892016384\,0 z^{35} + 5612222487551281257081916\,80 z^{34} + 17798695421\backslash$

$072697669468739680\, z^{33} + 432530168604725658189210596640\, z^{32} + 831518992033334153165\,8\backslash$

$617695280\, z^{31} + 129103723904595771409928232487740\, z^{30} + 163919073853198617069964709\,7\backslash$

$803790\, z^{29} + 1711104070984082303576075761868244\,0 z^{28} + 146518901443861803658771329\,86\backslash$

$6897880\, z^{27} + 101553427880666984315974532715125262\,0 z^{26} + 549633827605307607506875\,44\backslash$

$67310102760\, z^{25} + 2089057420971492753926706831574495164\,0 z^{24} + 301646099705919471898\backslash$

$27076234922007050\, z^{23} - 28912741628152937614209563101551926712\,0 z^{22} - 30536692798730\backslash$

$6334679315059193797470013\,0 z^{21} - 17566486109105161467894504789161406270600\, z^{20} -$

$736736506384615745386797430970500511152\,20 z^{19} - 24040743896755739891331729697533\backslash$

$6574702980\, z^{18} - 61916829368763951125106727397502019711410\,0 z^{17} - 12412253032904606\,2\backslash$

$37959908599059592265793\,20 z^{16} - 18355537951348376462623507792619318828947\,50 z^{15} -$

$16703146028371411108456407060310120735557\,00 z^{14} + 30668748618622961862750876075\,27\backslash$

$5710932000\, z^{13} + 293159415531339032893571618700161456826000\,0 z^{12} + 4919456458899666\,4\backslash$

$98684069708388548285600000\, z^{11} + 377736564624376265310479588420614333200000\,0 z^{10} +$

$6819563915441567451401786364159360000000\,0 z^{9} - 311881093752325372623566678266609\backslash$

$6800000000\, z^{8} - 37718337873996167042589088082942880000000\,00 z^{7} - 2488444831930996\,82\backslash$

$49089549891443200000000000\, z^{6} - 893959103422093803323305262745600000000000\,0 z^{5} - 1000\backslash$

$83705719332806676962561024000000000000000\, z^{4} + 256420308456773747641801728000000000\,0\backslash$

$0000\, z^{3} + 10280761031833373040014131200000000000000000\, z^{2} + 1033754008459758568243200\,0\backslash$

$00000000000000\, z)\, \mathrm{Dz}^{3} +$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{array}{l}
(219281667794999040 0\, z^{39} + 633490213477308768000\, z^{38} + 7286498601148445535360 0\, z^{37} + \\
48474868697955372605328 00\, z^{36} + 2170140170487616456148160 00\, z^{35} + 708801699580\backslash \\
0124707996090560\, z^{34} + 177409657131610270482016190640\, z^{33} + 3513089912736584156549 23\backslash \\
8676620\, z^{32} + 5620158773274044995967067545169 0\, z^{31} + 735847759326730529024504240 98\backslash \\
8015\, z^{30} + 793441106307331443298848248590050 0\, z^{29} + 704051719126302863599455718967 7\backslash \\
4110\, z^{28} + 508882813920850610699235633677324220\, z^{27} + 291338677229064650128281265554\backslash \\
6011475\, z^{26} + 1223798977496446338506289092696321595 0\, z^{25} + 273031638745556160521558 9\backslash \\
8475524386210\, z^{24} - 844007242726014050652717732643972095 30\, z^{23} - 130932954808576856 2\backslash \\
973129072537724164955\, z^{22} - 822926919906244244426426023497784780536 0\, z^{21} - \\
359487409188444751403185748400011152136 70\, z^{20} - 119246681320333134593914488403 3\backslash \\
28142970080\, z^{19} - 304273308297438630162837099041285050546455\, z^{18} - 577145030895901 79\backslash \\
069731112638689603676749 0\, z^{17} - 707778167790136602728038144967670916837350\, z^{16} - \\
153341406553907334245470038125935935813900\, z^{15} + 161120292062882594294021940651 58\backslash \\
76419542000\, z^{14} + 418899361620501704689973912421154454346000 0\, z^{13} + 569962639208201 8\backslash \\
0374532593961949063880000 00\, z^{12} + 414514018720330983618331139825246996400000 0\, z^{11} + \\
950688923971337731996303622505069600000000\, z^{10} - 322447694713681036240924354097202\backslash \\
9600000000\, z^{9} - 363683552813892830276766498739953600000000 0\, z^{8} - 210391871123659328\backslash \\
53731961115328000000000000\, z^{7} - 551590602425414384164117858252800000000000 0\, z^{6} + 1162\backslash \\
159256944109024208981780480000000000000\, z^{5} + 112838527684482371017981562880000000 00\backslash \\
0000\, z^{4} + 266404409116908139735547904000000000000000\, z^{3} + 242727262779317760737280 00\backslash \\
00000000000000\, z^{2})\, \mathrm{Dz}^{4} +
\end{array}
$$

Face-Centered Cubic Lattices [Koutschan 2013]

$(39072006261654374 4z^{40} + 1142166614243603074 56\,z^{39} + 134408223516159630696 96\,z^{38} +$

$917965180366474611870720\,z^{37} + 42237673932263775988570560\,z^{36} + 14182188393104\backslash$

$81932976078400\,z^{35} + 3648720683640800119768991 0640\,z^{34} + 742504152062765602237759 45\backslash$

$2720\,z^{33} + 12205694666919011642462560650930\,z^{32} + 1642510223944437789867637 3653\backslash$

$9405\,z^{31} + 182176711561283643405373740475505 4\,z^{30} + 166579383028240072672437243 6543\backslash$

$4191\,z^{29} + 124533460849620200009711445328730256\,z^{28} + 743593796442908540070532245488\backslash$

$378205\,z^{27} + 3336004607088107531634361889061221370\,z^{26} + 902022216185473608420262982\backslash$

$4390547047\,z^{25} - 919882472294340420544742129940427 7112\,z^{24} - 279449868802402514175 67\backslash$

$7041789492570017\,z^{23} - 19078634276619398855767231265989066437 90\,z^{22} - 85740836464757\backslash$

$100507575655426729796745 55\,z^{21} - 284055872968472310701836068565837708117 20\,z^{20} -$

$69574258175312955514440713973653616428745\,z^{19} - 1149914368924877116699378498249 1\backslash$

$2517430330\,z^{18} - 7037801720157986336449543216718272533367 5\,z^{17} + 261641966501089147 8\backslash$

$436560832161578428795 50\,z^{16} + 10494108247951363848374672098100255394 00000\,z^{15} + 2089\backslash$

$663780964272997600159898811800513390000\,z^{14} + 259786002679636380331331345092974504\backslash$

$0000000\,z^{13} + 1759136834585156085432113720072647266000000\,z^{12} - 1346756149912237131 0\backslash$

$874092829081128000000 0\,z^{11} - 157899609879137074628470745343916920000000 0\,z^{10} - 15568\backslash$

$11681322720025894531955998040000000000\,z^{9} - 678908917613499441761342434795200000 00\backslash$

$0000\,z^{8} + 314922159293504699010388187520000000000 0\,z^{7} + 1924073444597524252611218 33\backslash$

$472000000000000\,z^{6} + 10412684021044478206062895104000000000000 0\,z^{5} + 20982276643045\backslash$

$39359901286400000000000000 0\,z^{4} + 17874380983279966433280000000000000000 00\,z^{3})\,\mathrm{Dz}^{5} +$

Face-Centered Cubic Lattices [Koutschan 2013]

$$
\begin{aligned}
&(35882454730090752\,z^{41} + 10612604051614486656\,z^{40} + 1276532600942212775168\,z^{39} + \\
&89393980129433032096320\,z^{38} + 4221606838983473228197008\,z^{37} + 145494567985766484\backslash \\
&898923048\,z^{36} + 3840828004490920060950969480\,z^{35} + 80160062388267727172211985080\,z^{34} + \\
&1350855094398006902682870922050\,z^{33} + 18631082892630536824222949409585\,z^{32} + 2118\backslash \\
&157968344640547119736453 22142\,z^{31} + 1986708322085667572665525016037411\,z^{30} + 1526308\backslash \\
&2383031406770429022758762048\,z^{29} + 94068732852089205756130773605094705\,z^{28} + 4410553\backslash \\
&7622909592151335713091881133 8\,z^{27} + 1319636945498761264973744224282378779\,z^{26} - 1376\backslash \\
&268096732267953995912640790 41112\,z^{25} - 31072001737970299221405533198706303141\,z^{24} - \\
&226886176666918560987240200768631693150\,z^{23} - 1033954017266382248984767586852072\backslash \\
&344191\,z^{22} - 3356732946224373601649087937349109785896\,z^{21} - 757312621278500761889122\backslash \\
&5542456994124245\,z^{20} - 90764595394133031846417221347765738958 10\,z^{19} + 10278671248090\backslash \\
&33537740891835881540878842 5\,z^{18} + 851492743570432923859250336532942918535 50\,z^{17} + \\
&240689360358498296007939096187740586134000\,z^{16} + 429409878921957648790555775 2682\backslash \\
&42743350000\,z^{15} + 495779225046771906420255540348281344800000\,z^{14} + 28712136337931261\backslash \\
&68715623464844653780000 00\,z^{13} - 119682652007548350954457856750250720000000\,z^{12} - \\
&395683465592680867401293480616198000000000\,z^{11} - 327383462755042385949747691240 82\backslash \\
&4000000000\,z^{10} - 866425754505013910667872020195200000000000\,z^{9} + 597046839721706795 4\backslash \\
&8931977222400000000000\,z^{8} + 725116102774123909908393630720000000000 00\,z^{7} + 33882896\backslash \\
&75587207195688626176000000000000000\,z^{6} + 63111567713049173257666560000000000000000\,z^{5} + \\
&51232302181375699968000000000000000000000\,z^{4})\,\mathrm{Dz}^{6} +
\end{aligned}
$$

Face-Centered Cubic Lattices [Koutschan 2013]

$(1600200173148416\, z^{42} + 478782978712278912\, z^{41} + 58815380786135567104\, z^{40} + 4218590\backslash$
$040421804170816\, z^{39} + 204216444469816446653424\, z^{38} + 7214118624119937529541160\, z^{37} +$
$195106070712453547506798168\, z^{36} + 4168870319524368197533959000\, z^{35} + 7187441279\backslash$
$5312940511795668940\, z^{34} + 101352803991324920736784237827 0\, z^{33} + 11775924181048893848\backslash$
$357395670676\, z^{32} + 112862055818213392356279768225402\, z^{31} + 88634049483647556986674 11\backslash$
$39358344\, z^{30} + 5592675973186567437279733685351646\, z^{29} + 26983635759333711243427828 35\backslash$
$4079724\, z^{28} + 85059388463264142313662526542420618\, z^{27} + 2481695683318164448040331315\backslash$
$0735864\, z^{26} - 173973152992350329598479680652675 2758\, z^{25} - 1321568542142315740183390 3\backslash$
$137021991092\, z^{24} - 6010151473251777932954274989889345 3858\, z^{23} - 1874061219337400172 1\backslash$
$2741167478185137320\, z^{22} - 36708878673671506390841246216615651 5566\, z^{21} - 136331238303\backslash$
$9883490014154141815321463 40\, z^{20} + 2052937632229799753666758504303681446150\, z^{19} +$
$8942220864711302092023950168348534856300\, z^{18} + 2211277908345604739979169031967335 6\backslash$
$808000\, z^{17} + 366622998309648535483008954687235024800 00\, z^{16} + 38663936209054739955701\backslash$
$649076784708400000\, z^{15} + 155758412096326841847250746805511760 00000\, z^{14} - 23399775927\backslash$
$110778754739301057544560000000\, z^{13} - 4595758184455506810833869296180720000000 0\, z^{12} -$
$32525005285459811112066289505232000000000\, z^{11} - 366121829233752392904666430464000\backslash$
$0000000\, z^{10} + 109703953015066112928145371648000000000000\, z^{9} + 9713112405197935942595\backslash$
$5338240000000000000\, z^{8} + 4260978719420783892503777280000000000000\, z^{7} + 7564670791225\backslash$
$350266880000000000000000000\, z^{6} + 5920177140958969774080000000000000000000\, z^{5})\, \mathrm{Dz}^{7} +$

Face-Centered Cubic Lattices [Koutschan 2013]

$$
\begin{aligned}
&(27122036833024\,z^{43} + 8208413201024064\,z^{42} + 1028987679702510976\,z^{41} + 75518451\backslash\\
&137118783792\,z^{40} + 3743195619381989907184\,z^{39} + 135369638077546936261428\,z^{38} + 374561\backslash\\
&5314367420203992832\,z^{37} + 81811619367860049045984675\,z^{36} + 14404666372482039137743\backslash\\
&4250\,z^{35} + 2072433111304027502371917285 0\,z^{34} + 24544662754165204609779276821 4\,z^{33} +\\
&2395828801191215780780578117794\,z^{32} + 19147407470673111231862249418166\,z^{31} + 1228\backslash\\
&63963621496746370188659696702\,z^{30} + 602621255648485924378700672331054\,z^{29} + 19351926\backslash\\
&643016174761373376710883 60\,z^{28} + 6941527120367832642436442906732 34\,z^{27} - 39030042885\backslash\\
&81893545590128913387262 2\,z^{26} - 29764596280393319656487373367019177 4\,z^{25} - 1329742929\backslash\\
&7280072157040025492815915 38\,z^{24} - 39039896148256488192244326572087276 46\,z^{23} - 603801\backslash\\
&55340196640177774384173593119 14\,z^{22} + 7565280951156009750992823479550694170\,z^{21} +\\
&8332812633696018310177123954988378632 5\,z^{20} + 297859836972471180382017327162905 95\backslash\\
&5900\,z^{19} + 68122669439368525201713007390832584050 0\,z^{18} + 1055504316932586226613390 04\backslash\\
&4310017920000\,z^{17} + 97498262514411065483466068899043460000 0\,z^{16} + 809214817274247946\backslash\\
&231359306234720000 00\,z^{15} - 12456927789753712089804979366495800000 0\,z^{14} - 1877612972\backslash\\
&16604654284152589154800000000 0\,z^{13} - 11862016919810145440581800070800000000 0\,z^{12} -\\
&44241392637013980291878304000000000000 0\,z^{11} + 526588498855023559134177312000000 00\backslash\\
&0000\,z^{10} + 410999234738834010247469568000000000000 0\,z^{9} + 17433301221381095805118464 0\backslash\\
&00000000000 0\,z^{8} + 29950530354743793153638400000000000000 0\,z^{7} + 2276991208061142220 80\backslash\\
&000000000000000 0\,z^{6})\ \mathrm{Dz}^{8}
\end{aligned}
$$

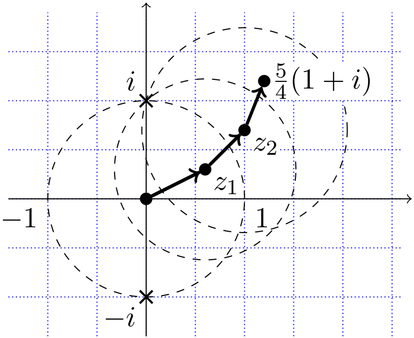
order 8, degree 43, 43-digit coefficients

Face-Centered Cubic Lattices [Koutschan 2013]

```
sage: mat = dop6.numerical_transition_matrix(
                                        [0, 3/2 + i, 1], 1e-60)

sage:  mat[0, 5]

   [1.0277491006274988398593636792739685020924399090011487 24252
   +/- 3.94e-58] + [+/- 1.16e-58]*I
```

(Wall time: $\approx$ 10 min)

How it Works: A Taylor Series Method



$\arctan\left(\frac{5}{4}(1+i)\right)$?

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\ldots + 0.22\ldots \\ 0 & 0.72\ldots - 0.20\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.36\ldots + 0.32\ldots \\ 0 & 0.75\ldots - 0.07\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

...

At each step, compute the sum of the power series expansion of each entry of the transition matrix.

The idea extends to the regular singular case.

How it Works: Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$

obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n)\, y_{n+s} + \cdots + b_1(n)\, y_{n+1} + b_0(n)\, y_n = 0.$$

(And conversely, for D-finite formal power series.)

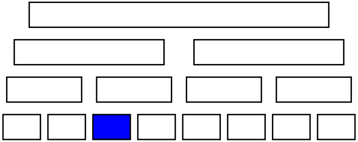
Leads to **fast algorithms** (not fully implemented yet)

[Schroeppel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988; van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

time $\boldsymbol{O(M(n \log^2 n))}$, space $\boldsymbol{O(n)}$

for fixed z and $\varepsilon = 2^{-n}$



How it Works: Error Bounds

Truncation Error

$$\sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{N-1} y_n z^n + \underbrace{\sum_{n=N}^{\infty} y_n z^n}_{?}$$

Majorants

[Cauchy; ...; van der Hoeven 2001; M. & Salvy 2010; M. 2016 ?]

- Bound the differential equation with a simple "**model equation**":

$$y'(z) = a(z)\, y(z) \qquad \lll \qquad g'(z) = \frac{1}{(1-\alpha z)}\, g(z)$$

- Solve the model equation and study the solutions:

$$\left| \sum_{k=n}^{+\infty} y_n z^n \right| \leqslant \sum_{k=n}^{+\infty} g_n |z|^n \leqslant ?$$

- ...Using the **residuals** $L(y_0 + \cdots + y_{n-1} z^{n-1})$ to obtain tight bounds

Summary

What it is: an extension of ore_algebra written in/for SageMath

What it does: numerical analytic continuation & singular connection, for arbitrary D-finite functions, with rigorous error bounds

How it works: Taylor series, analytic continuation, recurrences, majorants, ball arithmetic...

Code available at

http://marc.mezzarobba.net/code/ore_algebra-analytic

Perspectives

Fast algorithms, lower-level code,

evaluation on intervals, D-finite functions as objects,

irregular singular connection problems [van der Hoeven 2006]...

Comments, bug reports, feature requests, examples welcome!

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