The Dynamic Dictionary of Mathematical Functions

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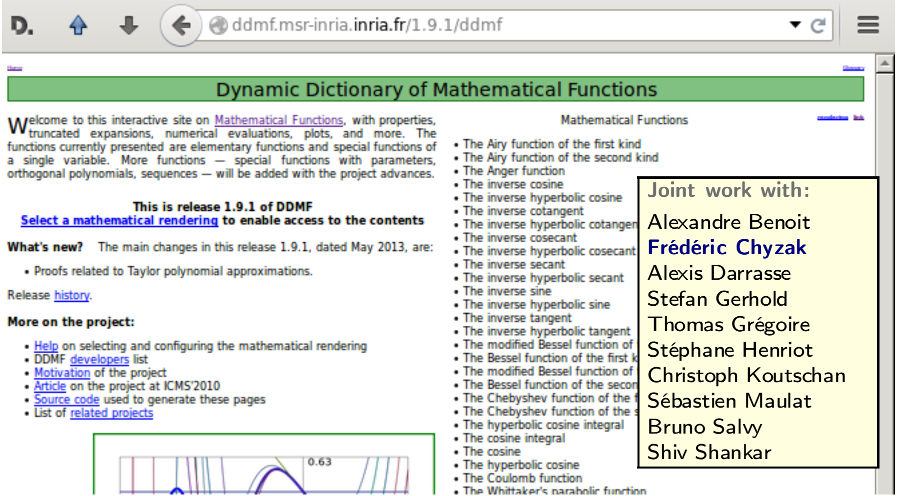
1 The DDMF

The DDMF Project

(∼2009–2012)


ddmf.msr-inria.inria.fr/1.9.1/ddmf

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

This is release 1.9.1 of DDMF
Select a mathematical rendering to enable access to the contents

What's new? The main changes in this release 1.9.1, dated May 2013, are:

- Proofs related to Taylor polynomial approximations.

Release history.

More on the project:

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project
- Article on the project at ICMS'2010
- Source code used to generate these pages
- List of related projects

0.63

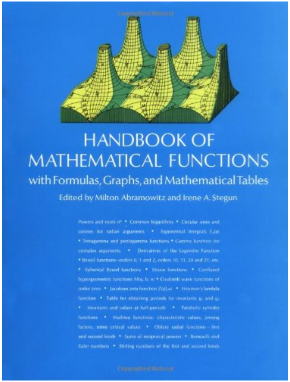
Mathematical Functions

- The Airy function of the first kind
- The Airy function of the second kind
- The Anger function
- The inverse cosine
- The inverse hyperbolic cosine
- The inverse cotangent
- The inverse hyperbolic cotangen
- The inverse cosecant
- The inverse hyperbolic cosecant
- The inverse secant
- The inverse hyperbolic secant
- The inverse sine
- The inverse hyperbolic sine
- The inverse tangent
- The inverse hyperbolic tangent
- The modified Bessel function of
- The Bessel function of the first k
- The modified Bessel function of
- The Bessel function of the secon
- The Chebyshev function of the f
- The Chebyshev function of the s
- The hyperbolic cosine integral
- The cosine integral
- The cosine
- The hyperbolic cosine
- The Coulomb function
- The Whittaker's parabolic function

Joint work with:

Alexandre Benoit
Frédéric Chyzak
Alexis Darrasse
Stefan Gerhold
Thomas Grégoire
Stéphane Henriot
Christoph Koutschan
Sébastien Maulat
Bruno Salvy
Shiv Shankar

An Encyclopedia of Special Functions



Ascending Series

10.4.2 $\text{Ai}(z) = c_1 f(z) - c_2 g(z)$

10.4.3 $\text{Bi}(z) = \sqrt{3}\,[c_1 f(z) + c_2 g(z)]$

$$f(z) = 1 + \frac{1}{3!} z^3 + \frac{1\cdot 4}{6!} z^6 + \frac{1\cdot 4\cdot 7}{9!} z^9 + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!}$$

$$g(z) = z + \frac{2}{4!} z^4 + \frac{2\cdot 5}{7!} z^7 + \frac{2\cdot 5\cdot 8}{10!} z^{10} + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$\left(\alpha + \frac{1}{3}\right)_0 = 1$$

$$3^k \left(\alpha + \frac{1}{3}\right)_k = (3\alpha+1)(3\alpha+4)\dots(3\alpha+3k-2)$$

$(\alpha$ arbitrary; $k = 1, 2, 3, \dots)$

(See **6.1.22.**)

10.4.4

$c_1 = \text{Ai}(0) = \text{Bi}(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3)$

$= .35502\ 80538\ 87817$

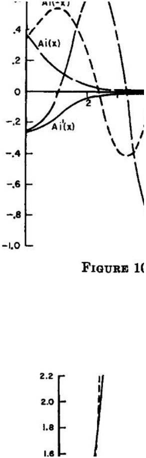


FIGURE 1(

An Encyclopedia of Special Functions

Representations in Terms of Bessel Functions

$$\zeta=\tfrac{2}{3}z^{3/2}$$

10.4.14
$$\mathrm{Ai}\,(z)=\tfrac{1}{3}\sqrt{z}[I_{-1/3}(\zeta)-I_{1/3}(\zeta)]=\pi^{-1}\sqrt{z/3}K_{1/3}(\zeta)$$

10.4.15
$$\mathrm{Ai}\,(-z)=\tfrac{1}{3}\sqrt{z}[J_{1/3}(\zeta)+J_{-1/3}(\zeta)]$$
$$=\tfrac{1}{2}\sqrt{z/3}[e^{\pi i/6}H^{(1)}_{1/3}(\zeta)+e^{-\pi i/6}H^{(2)}_{1/3}(\zeta)]$$

10.4.16
$$^{*}\ -\mathrm{Ai}'(z)=\tfrac{1}{3}z[I_{-2/3}(\zeta)-I_{2/3}(\zeta)]=\pi^{-1}(z/\sqrt{3})K_{2/3}(\zeta)$$

10.4.17
$$\mathrm{Ai}'\,(-z)=-\tfrac{1}{3}z[J_{-2/3}(\zeta)-J_{2/3}(\zeta)]$$
$$=\tfrac{1}{2}(z/\sqrt{3})[e^{-\pi i/6}H^{(1)}_{2/3}(\zeta)+e^{\pi i/6}H^{(2)}_{2/3}(\zeta)]$$

10.4.18 $$\mathrm{Bi}\,(z)=\sqrt{z/3}[I_{-1/3}(\zeta)+I_{1/3}(\zeta)]$$

10.4.19
$$\mathrm{Bi}\,(-z)=\sqrt{z/3}[J_{-1/3}(\zeta)-J_{1/3}(\zeta)]$$
$$=\tfrac{1}{2}i\sqrt{z/3}[e^{\pi i/6}H^{(1)}_{1/3}(\zeta)-e^{-\pi i/6}H^{(2)}_{1/3}(\zeta)]$$

10.4.20 $$\mathrm{Bi}'\,(z)=(z/\sqrt{3})[I_{-2/3}(\zeta)+I_{2/3}(\zeta)]$$

10.4.21
$$\mathrm{Bi}'\,(-z)=(z/\sqrt{3})[J_{-2/3}(\zeta)+J_{2/3}(\zeta)]$$
$$=\tfrac{1}{2}i(z/\sqrt{3})[e^{-\pi i/6}H^{(1)}_{2/3}(\zeta)-e^{\pi i/6}H^{(2)}_{2/3}(\zeta)]$$

Representations of Bessel Functions in Terms of Airy Functions

$$z=\left(\frac{3}{2}\zeta\right)^{2/3}$$

10.4.22 $$J_{\pm1/3}(\zeta)=\tfrac{1}{2}\sqrt{3/z}[\sqrt{3}\,\mathrm{Ai}\,(-z)\mp\mathrm{Bi}\,(-z)]$$

***10.4.23** $$H^{(1)}_{\pm1/3}(\zeta)=e^{\mp\pi i/6}\sqrt{3/z}[\mathrm{Ai}\,(-z)-i\,\mathrm{Bi}\,(-z)]$$

10.4.24 $$H^{(2)}_{\pm1/3}(\zeta)=e^{\pm\pi i/6}\sqrt{3/z}[\mathrm{Ai}\,(-z)+i\,\mathrm{Bi}\,(-z)]$$

10.4.25 $$I_{\pm1/3}(\zeta)=\tfrac{1}{2}\sqrt{3/z}[\mp\sqrt{3}\,\mathrm{Ai}\,(z)+\mathrm{Bi}\,(z)]$$

10.4.26 $$K_{\pm1/3}(\zeta)=\pi\sqrt{3/z}\,\mathrm{Ai}\,(z)$$

10.4.27 $$J_{\pm2/3}(\zeta)=(\sqrt{3}/2z)[\pm\sqrt{3}\,\mathrm{Ai}'\,(-z)+\mathrm{Bi}'\,(-z)]$$

10.4.28
$$H^{(1)}_{2/3}(\zeta)=e^{-2\pi i/3}H^{(1)}_{-2/3}(\zeta)$$
$$=e^{\pi i/6}(\sqrt{3}/z)[\mathrm{Ai}'\,(-z)-i\,\mathrm{Bi}'\,(-z)]$$

10.4.29
$$H^{(2)}_{2/3}(\zeta)=e^{2\pi i/3}H^{(2)}_{-2/3}(\zeta)$$
$$=e^{-\pi i/6}(\sqrt{3}/z)[\mathrm{Ai}'\,(-z)+i\,\mathrm{Bi}'\,(-z)]$$

*See page II.

10.4.30 $$I_{\pm2/3}(\zeta)=(\sqrt{3}/2z)[\pm\sqrt{3}\,\mathrm{Ai}'\,(z)+\mathrm{Bi}'\,(z)]$$

10.4.31 $$K_{\pm2/3}(\zeta)=-\pi(\sqrt{3}/z)\,\mathrm{Ai}'\,(z)$$

Integral Representations

10.4.32
$$(3a)^{-1/3}\pi\,\mathrm{Ai}\,[\pm(3a)^{-1/3}x]=\int_0^\infty\cos\,(at^3\pm xt)dt$$

10.4.33
$$(3a)^{-1/3}\pi\,\mathrm{Bi}\,[\pm(3a)^{-1/3}x]$$
$$=\int_0^\infty[\exp\,(-at^3\pm xt)+\sin\,(at^3\pm xt)]dt$$

The Integrals $\int_0^z\mathrm{Ai}\,(\pm t)dt$, $\int_0^z\mathrm{Bi}\,(\pm t)dt$

$$\zeta=\tfrac{2}{3}z^{3/2}$$

10.4.34 $$\int_0^z\mathrm{Ai}\,(t)dt=\frac{1}{3}\int_0^\zeta[I_{-1/3}(t)-I_{1/3}(t)]dt$$

10.4.35 $$\int_0^z\mathrm{Ai}\,(-t)dt=\frac{1}{3}\int_0^\zeta[J_{-1/3}(t)+J_{1/3}(t)]dt$$

10.4.36 $$\int_0^z\mathrm{Bi}\,(t)dt=\frac{1}{\sqrt{3}}\int_0^\zeta[I_{-1/3}(t)+I_{1/3}(t)]dt$$

10.4.37 $$\int_0^z\mathrm{Bi}\,(-t)dt=\frac{1}{\sqrt{3}}\int_0^\zeta[J_{-1/3}(t)-J_{1/3}(t)]dt$$

Ascending Series for $\int_0^z\mathrm{Ai}\,(\pm t)dt$, $\int_0^z\mathrm{Bi}\,(\pm t)dt$

10.4.38 $$\int_0^z\mathrm{Ai}\,(t)dt=c_1F(z)-c_2G(z)$$

(See **10.4.2.**)

10.4.39 $$\int_0^z\mathrm{Ai}\,(-t)dt=-c_1F(-z)+c_2G(-z)$$

10.4.40 $$\int_0^z\mathrm{Bi}\,(t)dt=\sqrt{3}[c_1F(z)+c_2G(z)]$$

(See **10.4.3.**)

10.4.41
$$\int_0^z\mathrm{Bi}\,(-t)dt=-\sqrt{3}[c_1F(-z)+c_2G(-z)]$$
$$F(z)=z+\frac{1}{4!}z^4+\frac{1\cdot4}{7!}z^7+\frac{1\cdot4\cdot7}{10!}z^{10}+\dots$$
$$=\sum_0^\infty3^k\left(\frac{1}{3}\right)_k\frac{z^{3k+1}}{(3k+1)!}$$
$$G(z)=\frac{1}{2!}z^2+\frac{2}{5!}z^5+\frac{2\cdot5}{8!}z^8+\frac{2\cdot5\cdot8}{11!}z^{11}+\dots$$
$$=\sum_0^\infty3^k\left(\frac{2}{3}\right)_k\frac{z^{3k+2}}{(3k+2)!}$$

The constants c_1, c_2 are given in **10.4.4, 10.4.5.**

AIRY FUNCTIONS **Table 10.11**

x	Ai(x)	Ai′(x)	Bi(x)	Bi′(x)	x	Ai(x)	Ai′(x)	Bi(x)	Bi′(x)
0.00	0.35502 805	−0.25881 940	0.61492 663	0.44828 836	0.50	0.23169 361	−0.22491 053	0.85427 704	0.54457 256
0.01	0.35243 992	−0.25880 174	0.61940 962	0.44831 926	0.51	0.22945 031	−0.22374 617	0.85974 431	0.54890 049
0.02	0.34985 214	−0.25874 909	0.62389 322	0.44841 254	0.52	0.22721 872	−0.22257 027	0.86525 543	0.55334 239
0.03	0.34726 505	−0.25866 197	0.62837 808	0.44856 911	0.53	0.22499 894	−0.22138 322	0.87081 154	0.55789 959
0.04	0.34467 901	−0.25854 090	0.63286 482	0.44878 987	0.54	0.22279 109	−0.22018 541	0.87641 381	0.56257 345
0.05	0.34209 435	−0.25838 640	0.63735 409	0.44907 570	0.55	0.22059 527	−0.21897 720	0.88206 341	0.56736 532
0.06	0.33951 139	−0.25819 898	0.64184 655	0.44942 752	0.56	0.21841 158	−0.21775 898	0.88776 152	0.57227 662
0.07	0.33693 047	−0.25797 916	0.64634 286	0.44984 622	0.57	0.21624 012	−0.21653 112	0.89350 934	0.57730 873
0.08	0.33435 191	−0.25772 745	0.65084 370	0.45033 270	0.58	0.21408 099	−0.21529 397	0.89930 810	0.58246 311
0.09	0.33177 603	−0.25744 437	0.65534 975	0.45088 787	0.59	0.21193 427	−0.21404 790	0.90515 902	0.58774 120
0.10	0.32920 313	−0.25713 042	0.65986 169	0.45151 263	0.60	0.20980 006	−0.21279 326	0.91106 334	0.59314 448
0.11	0.32663 352	−0.25678 613	0.66438 023	0.45220 789	0.61	0.20767 844	−0.21153 041	0.91702 233	0.59867 447
0.12	0.32406 751	−0.25641 200	0.66890 609	0.45297 457	0.62	0.20556 948	−0.21025 970	0.92303 726	0.60433 267
0.13	0.32150 538	−0.25600 854	0.67343 997	0.45381 357	0.63	0.20347 327	−0.20898 146	0.92910 941	0.61012 064
0.14	0.31894 743	−0.25557 625	0.67798 260	0.45472 582	0.64	0.20138 987	−0.20769 605	0.93524 011	0.61603 997
0.15	0.31639 395	−0.25511 565	0.68253 473	0.45571 223	0.65	0.19931 937	−0.20640 378	0.94143 066	0.62209 226
0.16	0.31384 521	−0.25462 724	0.68709 709	0.45677 373	0.66	0.19726 182	−0.20510 500	0.94768 241	0.62827 912
0.17	0.31130 150	−0.25411 151	0.69167 046	0.45791 125	0.67	0.19521 729	−0.20380 004	0.95399 670	0.63460 222
0.18	0.30876 307	−0.25356 898	0.69625 558	0.45912 572	0.68	0.19318 584	−0.20248 920	0.96037 491	0.64106 324
0.19	0.30623 020	−0.25300 013	0.70085 323	0.46041 808	0.69	0.19116 752	−0.20117 281	0.96681 843	0.64766 389
0.20	0.30370 315	−0.25240 547	0.70546 420	0.46178 928	0.70	0.18916 240	−0.19985 119	0.97332 866	0.65440 592
0.21	0.30118 218	−0.25178 548	0.71008 928	0.46324 026	0.71	0.18717 052	−0.19852 464	0.97990 703	0.66129 109
0.22	0.29866 753	−0.25114 067	0.71472 927	0.46477 197	0.72	0.18519 192	−0.19719 347	0.98655 496	0.66832 121
0.23	0.29615 945	−0.25047 151	0.71938 499	0.46638 539	0.73	0.18322 666	−0.19585 798	0.99327 394	0.67549 810
0.24	0.29365 818	−0.24977 850	0.72405 726	0.46808 147	0.74	0.18127 478	−0.19451 846	1.00006 542	0.68282 363
0.25	0.29116 395	−0.24906 211	0.72874 690	0.46986 119	0.75	0.17933 631	−0.19317 521	1.00693 091	0.69029 970
0.26	0.28867 701	−0.24832 284	0.73345 477	0.47172 554	0.76	0.17741 128	−0.19182 851	1.01387 192	0.69792 824
0.27	0.28619 757	−0.24756 115	0.73818 170	0.47367 549	0.77	0.17549 975	−0.19047 865	1.02088 999	0.70571 121
0.28	0.28372 586	−0.24677 753	0.74292 857	0.47571 205	0.78	0.17360 172	−0.18912 591	1.02798 667	0.71365 062
0.29	0.28126 209	−0.24597 244	0.74769 624	0.47783 623	0.79	0.17171 724	−0.18777 055	1.03516 353	0.72174 849
0.30	0.27880 648	−0.24514 636	0.75248 559	0.48004 903	0.80	0.16984 632	−0.18641 286	1.04242 217	0.73000 690
0.31	0.27635 923	−0.24429 976	0.75729 752	0.48235 148	0.81	0.16798 899	−0.18505 310	1.04976 421	0.73842 795
0.32	0.27392 055	−0.24343 309	0.76213 292	0.48474 462	0.82	0.16614 526	−0.18369 153	1.05719 128	0.74701 380
0.33	0.27149 064	−0.24254 682	0.76699 272	0.48722 948	0.83	0.16431 516	−0.18232 840	1.06470 504	0.75576 663
0.34	0.26906 968	−0.24164 140	0.77187 782	0.48980 713	0.84	0.16249 870	−0.18096 398	1.07230 717	0.76468 865
0.35	0.26665 787	−0.24071 730	0.77678 917	0.49247 861	0.85	0.16069 588	−0.17959 851	1.07999 939	0.77378 215
0.36	0.26425 540	−0.23977 495	0.78172 770	0.49524 501	0.86	0.15890 673	−0.17823 223	1.08778 340	0.78304 942
0.37	0.26186 243	−0.23881 481	0.78669 439	0.49810 741	0.87	0.15713 124	−0.17686 539	1.09566 096	0.79249 282
0.38	0.25947 916	−0.23783 731	0.79169 018	0.50106 692	0.88	0.15536 942	−0.17549 823	1.10363 385	0.80211 473
0.39	0.25710 574	−0.23684 291	0.79671 605	0.50412 463	0.89	0.15362 128	−0.17413 097	1.11170 386	0.81191 759
0.40	0.25474 235	−0.23583 203	0.80177 300	0.50728 168	0.90	0.15188 680	−0.17276 384	1.11987 281	0.82190 389
0.41	0.25238 916	−0.23480 512	0.80686 202	0.51053 920	0.91	0.15016 600	−0.17139 708	1.12814 255	0.83207 615
0.42	0.25004 630	−0.23376 259	0.81198 412	0.51389 833	0.92	0.14845 886	−0.17003 090	1.13651 496	0.84243 695
0.43	0.24771 395	−0.23270 487	0.81714 033	0.51736 025	0.93	0.14676 538	−0.16866 551	1.14499 193	0.85298 891
0.44	0.24539 226	−0.23163 239	0.82233 167	0.52092 614	0.94	0.14508 555	−0.16730 113	1.15357 539	0.86373 470
0.45	0.24308 135	−0.23054 556	0.82755 920	0.52459 717	0.95	0.14341 935	−0.16593 797	1.16226 728	0.87467 704
0.46	0.24078 139	−0.22944 479	0.83282 397	0.52837 457	0.96	0.14176 678	−0.16457 623	1.17106 959	0.88581 871
0.47	0.23849 250	−0.22833 050	0.83812 705	0.53225 956	0.97	0.14012 782	−0.16321 611	1.17998 433	0.89716 253
0.48	0.23621 482	−0.22720 310	0.84346 952	0.53625 338	0.98	0.13850 245	−0.16185 781	1.18901 352	0.90871 137
0.49	0.23394 848	−0.22606 297	0.84885 248	0.54035 729	0.99	0.13689 066	−0.16050 153	1.19815 925	0.92046 818
0.50	0.23169 361	−0.22491 053	0.85427 704	0.54457 256	1.00	0.13529 242	−0.15914 744	1.20742 359	0.93243 593
	$\left[{(-6)1 \atop 4}\right]$	$\left[{(-6)4 \atop 4}\right]$	$\left[{(-6)5 \atop 4}\right]$	$\left[{(-5)1 \atop 4}\right]$		$\left[{(-6)2 \atop 4}\right]$	$\left[{(-6)1 \atop 4}\right]$	$\left[{(-5)1 \atop 4}\right]$	$\left[{(-5)3 \atop 4}\right]$

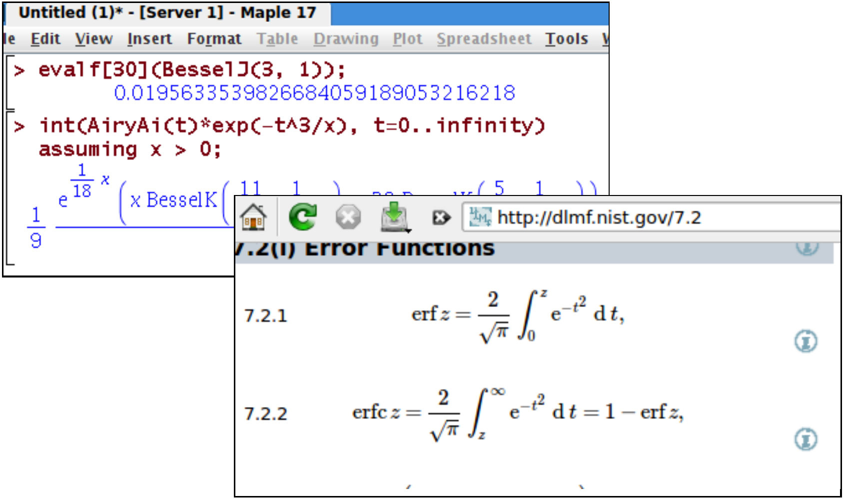
AIRY FUNCTIONS—AUXILIARY FUNCTIONS FOR LARGE POSITIVE ARGUMENTS

ζ^{-1}	x	$f(-\zeta)$	$f(\zeta)$	$g(-\zeta)$	$g(\zeta)$	ζ^{-1}	x	$f(-\zeta)$	$f(\zeta)$	$g(-\zeta)$	$g(\zeta)$
1.5	1.000000	0.527027	0.619912	0.619954	0.478728	0.50	2.080084	0.548230	0.593015	0.587245	0.526011
1.4	1.047069	0.528783	0.620335	0.617156	0.479925	0.45	2.231443	0.549584	0.589451	0.585235	0.530678
1.3	1.100099	0.530601	0.620327	0.614275	0.481658	0.40	2.413723	0.550980	0.585855	0.583174	0.535345
1.2	1.160397	0.532488	0.619799	0.611305	0.484018	0.35	2.638450	0.552421	0.582330	0.581056	0.539902
1.1	1.229700	0.534448	0.618649	0.608239	0.487107	0.30	2.924018	0.553912	0.578985	0.578878	0.544235
1.0	1.310371	0.536489	0.616764	0.605068	0.491037	0.25	3.301927	0.555456	0.575908	0.576635	0.548255
0.9	1.405721	0.538618	0.614022	0.601782	0.495921	0.20	3.831547	0.557058	0.573135	0.574320	0.551930
0.8	1.520550	0.540844	0.610309	0.598372	0.501859	0.15	4.641589	0.558724	0.570636	0.571927	0.555296
0.7	1.662119	0.543180	0.605543	0.594823	0.508909	0.10	6.082202	0.560462	0.568343	0.569448	0.558428
0.6	1.842016	0.545636	0.599723	0.591120	0.517032	0.05	9.654894	0.562280	0.566204	0.566873	0.561382
0.5	2.080084	0.548230	0.593015	0.587245	0.526011	0.00	∞	0.564190	0.564190	0.564190	0.564190
	$\left[{(-3)7 \atop 9}\right]$	$\left[{(-5)2 \atop 4}\right]$	$\left[{(-4)1 \atop 6}\right]$	$\left[{(-5)2 \atop 4}\right]$	$\left[{(-4)1 \atop 6}\right]$			$\left[{(-5)1 \atop 4}\right]$	$\left[{(-5)4 \atop 6}\right]$	$\left[{(-5)1 \atop 4}\right]$	$\left[{(-5)4 \atop 6}\right]$

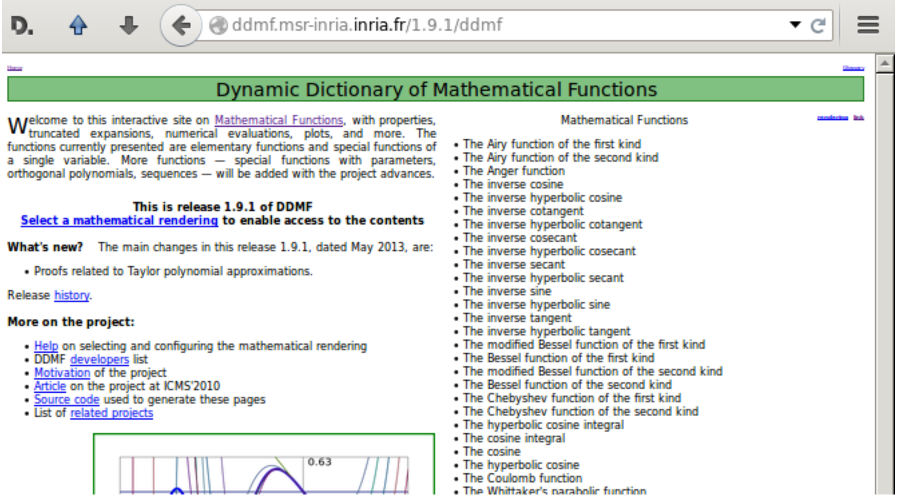
$\mathrm{Ai}(x)\sim\frac{1}{2}x^{-\frac{1}{4}}e^{-\zeta}f(-\zeta)$ $\quad\mathrm{Bi}(x)\sim x^{-\frac{1}{4}}e^{\zeta}f(\zeta)$ $\quad\mathrm{Ai}'(x)\sim-\frac{1}{2}x^{\frac{1}{4}}e^{-\zeta}g(-\zeta)$ $\quad\mathrm{Bi}'(x)\sim x^{\frac{1}{4}}e^{\zeta}g(\zeta)$ $\quad\zeta=\frac{2}{3}x^{\frac{3}{2}}$

From J. C. P. Miller, The Airy integral, British Assoc. Adv. Sci. Mathematical Tables, Part-vol. B. Cambridge Univ. Press, Cambridge, England, 1946 (with permission).

An Encyclopedia of Special Functions



Demo!



ddmf.msr-inria.inria.fr/1.9.1/ddmf

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

This is release 1.9.1 of DDMF
Select a mathematical rendering to enable access to the contents

What's new? The main changes in this release 1.9.1, dated May 2013, are:

- Proofs related to Taylor polynomial approximations.

Release history.

More on the project:

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project
- Article on the project at ICMS'2010
- Source code used to generate these pages
- List of related projects

Mathematical Functions

- The Airy function of the first kind
- The Airy function of the second kind
- The Anger function
- The inverse cosine
- The inverse hyperbolic cosine
- The inverse cotangent
- The inverse hyperbolic cotangent
- The inverse cosecant
- The inverse hyperbolic cosecant
- The inverse secant
- The inverse hyperbolic secant
- The inverse sine
- The inverse hyperbolic sine
- The inverse tangent
- The inverse hyperbolic tangent
- The modified Bessel function of the first kind
- The Bessel function of the first kind
- The modified Bessel function of the second kind
- The Bessel function of the second kind
- The Chebyshev function of the first kind
- The Chebyshev function of the second kind
- The hyperbolic cosine integral
- The cosine integral
- The cosine
- The hyperbolic cosine
- The Coulomb function
- The Whittaker's parabolic function

Behind the Scenes...

- **DynaMoW** — Ocaml + Maple + web [Chyzak & Darrasse]

- **Algolib**

  http://algo.inria.fr/libraries/

  GNU LGPL

  - **gfun** — D-finite power series [Salvy & Zimmermann]
  - **MGfun** — Multivariate D-finite functions [Chyzak]
  - **NumGfun** — D-finite analytic functions [M.]
  - ...

- (+ DDMF-specific Maple libraries)

2 Identities

D-Finite Functions

[Stanley 1980, ...]

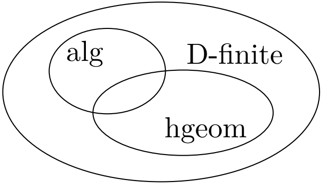
An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

$$a_r(z)\,y^{(r)}(z) + \cdots + a_1(z)y'(z) + a_0(z)\,y(z) = 0, \qquad a_j \in \mathbb{K}[z]$$

Examples: $\mathbf{sin}$ "=" $\{\boldsymbol{y''(z)} + \boldsymbol{y(z)} = 0, \quad y(0) = 0, \quad y'(0) = 1\}$

$\boldsymbol{K_0}$ "=" $\{z\,\boldsymbol{y''(z)} + \boldsymbol{y'(z)} - z\,\boldsymbol{y(z)} = 0, \quad (\text{ini?})\}$

More examples: exp, ln, Ai, erf, J_ν, algebraic fns, hypergeometric fns...



Not D-finite: ζ, Γ, W, tan, $C_{a,q}$, ...

D-Finite Functions

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Equivalently:

$$\dim_{\mathbb{K}(z)} \operatorname{Span}_{\mathbb{K}(z)} (D^i \cdot y)_{i \in \mathbb{N}} < \infty$$

$$\begin{aligned}
& \boldsymbol{y} \\
D\ \cdot y &= \boldsymbol{y'} \\
D^2 \cdot y &= \boldsymbol{y''} \\
D^3 \cdot y &= -\frac{a_0}{a_3}\,\boldsymbol{y} - \frac{a_1}{a_3}\,\boldsymbol{y'} - \frac{a_2}{a_1}\,\boldsymbol{y''} \\
D^4 \cdot y &= *\boldsymbol{y} + *\boldsymbol{y'} + *\boldsymbol{y''} \\
&\vdots
\end{aligned}$$

Closure Properties

D-finite functions over $\mathbb{K}$ form a $\mathbb{K}$-algebra.

Example:

$f(z) = \sin z$

$\boldsymbol{f''(z)} + \boldsymbol{f(z)} = 0$

$g(z) = \mathrm{Ai}(z)$

$\boldsymbol{g''(z)} - z\,\boldsymbol{g(z)} = 0$

$$fg = \boldsymbol{fg}$$

$$(fg)' = \boldsymbol{f'g} + \boldsymbol{fg'}$$

$$\begin{aligned}(fg)'' &= f''g + f'g' + f'g' + fg'' \\ &= -fg + 2f'g' + zfg \\ &= (z-1)\,\boldsymbol{fg} + 2\,\boldsymbol{f'g'}\end{aligned}$$

$$(fg)^{(3)} = \boldsymbol{fg} + (3z-1)\,\boldsymbol{f'g} + (z-3)\,\boldsymbol{fg'}$$

$$\begin{aligned}(fg)^{(4)} &= (z^2-6z+1)\,\boldsymbol{fg} + 4\,\boldsymbol{f'g} \\ &\quad + 2\,\boldsymbol{fg'} + (4z-4)\,\boldsymbol{f'g'}\end{aligned}$$

5 vectors

dimension **4**

$$(z+1)\,(fg)^{(4)} - (fg)^{(3)} - 2\,(z^2+1)\,(fg)'' - (z+5)\,(fg)' + (z^3+\cdots)\,(fg) = 0$$

Proof of Identities

Proposition. $\cos^2 z + \sin^2 z = 1.$

Proof. $\cos^2 z + \sin^2 z - 1 = \mathrm{O}(z^9).$ □

Why is this a proof?

- $\cos$ and $\sin$ both satisfy equations of order 2
- hence, $\cos^2$ and $\sin^2$ satisfy equations of order $\leqslant 4$
- 1 satisfies $y' = 0$
- hence, $\cos^2 + \sin^2 - 1$ satisfies an equation of order $\leqslant 9$

(Actually, $\cos^2 x$, $\sin^2 x$ and 1 all satisfy $y^{(3)} + 4\,y' = 0$.)

In CS terms: equality is decidable.

Generalisations

[Lipshitz, Zeilberger, Chyzak & Salvy, ...]

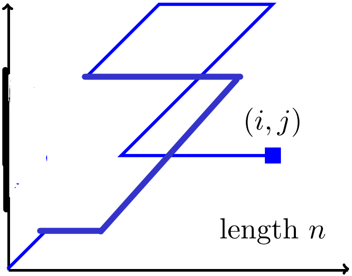
$$T_n(x) \qquad (-1)^{m-k}\,k!\binom{n-k}{m-k}\left\{{n+1 \atop k+1}\right\} \qquad \frac{\cos(z\,t)}{\sqrt{1-t^2}}$$

$$\binom{n}{k} \qquad \frac{\exp\left(\dfrac{4\,u\,(x\,y-u\,(x^2+y^2))}{1-4\,u^2}\right)}{\sqrt{1-4\,u^2}} \qquad \binom{m}{k}B_{n+k}$$

$$\frac{e^{-px}\,T_n(x)}{\sqrt{1-x^2}} \qquad {}_2F_1\left(\left.{a \quad b \atop a+b+1/2}\right|\, z\right) \qquad 2^n$$

$$\{T_n(x)\} \quad \text{“=“} \quad \begin{cases}(1-x^2)\,T_n''(x)-x\,T_n'(x)+n^2\,T_n(x)=0\\ n\,T_{n+1}(x)+2\,(1-x^2)\,T_n'(x)-n\,T_{n-1}(x)=0\\ \text{(+ initial values)}\end{cases}$$

Gessel Walks



Allowed steps: $\rightarrow, \leftarrow, \nearrow, \swarrow$

$$Q(x,y,t) = \sum_{n,i,j} \boldsymbol{q_{i,j,n}}\, x^i y^j t^n$$

Theorem. ("Gessel's conjecture")

[Kauers-Koutschan-Zeilberger 2008]

$$Q(0,0,t) = {}_3F_2\left(\left.\begin{matrix} 5/6,\ 1/2,\ 1 \\ 5/3,\ 2 \end{matrix}\,\right|\, 16\,t^2\right).$$

Theorem. [Bostan-Kauers 2010]

$Q(x,y,t)$ is algebraic.

(Estimated size of min. poly: 30 Gb.)

Needs fast algorithms!

Guessing

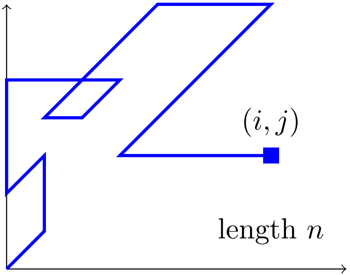
Given the **first terms** of a series $y(z) = y_0 + y_1 z + y_2 z^2 + \cdots$,
find **a differential operator L** of order $< r$,
with polynomial coefficients of degree $< d$,
such that $\boldsymbol{L \cdot y(z) = O(z^\sigma)}$. $\longrightarrow$ That's just linear algebra

Guess-and-prove

Faced with a conjectured identity $A(z) = B(z)$:

1. **compute** the initial terms of the series expansions of A and B;
2. **guess** a common differential equation;
3. **verify** that it is satisfied and check a finite number of initial values.

Gessel Walks



Allowed steps: $\rightarrow, \leftarrow, \nearrow, \swarrow$

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Needs fast algorithms!

3 Approximations

Differential Equations and Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$

obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n)\, y_{n+s} + \cdots + b_1(n)\, y_{n+1} + b_0(n)\, y_n = 0.$$

(And conversely, for D-finite formal power series.)

Example. $y'' + y = 0 \quad \Rightarrow \quad (n+1)\,(n+1)\, y_{n+2} + y_n = 0$

More recurrences: Generalized Fourier series. $y(x) = \sum_{n \in \mathbb{Z}} c_n T_n(x)$

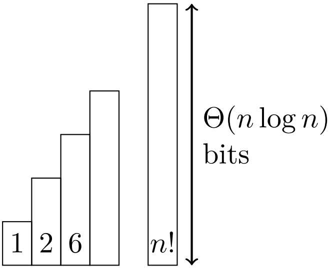
$$c_{n-3} + (2\,n-1)\,(2\,n-5)\,c_{n-1} - (2\,n+1)\,(2\,n+5)\,c_{n+1} - c_{n+3} = 0$$

Recurrences Lead to Fast Algorithms

First n terms	$O(n)$ ring operations
nth term	$\tilde{O}(\sqrt{n})$ ring operations (“baby steps-giant steps”)
nth term, over $\mathbb{Q}$	$O(M(n\log^2 n))$ bit operations (“binary splitting”)

Naïve algorithm

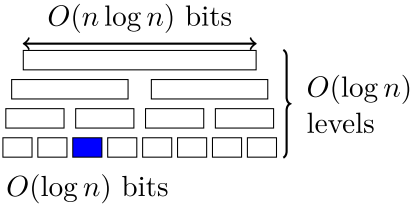
$$n! = n \cdot (n-1)!$$



Time: $\Omega(n^2 \log n)$

Binary splitting

$$n! = (n \cdots (\lfloor n/2 \rfloor + 1)) \cdot \lfloor n/2 \rfloor !$$



Time: $O(M(n \log n) \log n)$

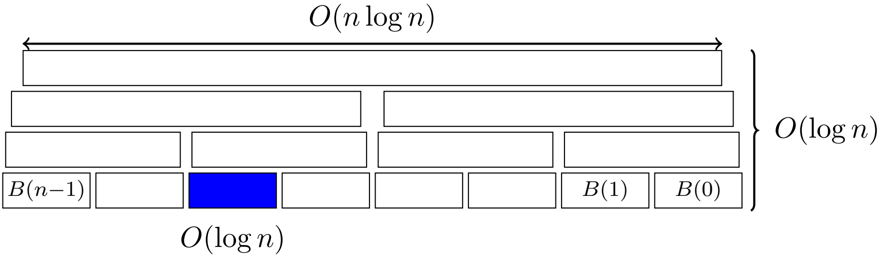
Binary Splitting for D-Finite Series

Same idea, using a matrix product tree:

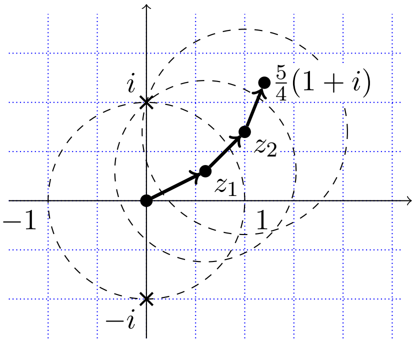
$$y(z) = \sum_{n=0}^{\infty} y_n \, z^n$$

$$\boldsymbol{S_n} = \sum_{k=0}^{n-1} y_k \, \zeta^k$$

$$\begin{bmatrix} y_{n+1} \\ \vdots \\ y_{n+s} \\ \boldsymbol{S_{n+1}} \end{bmatrix} = \begin{bmatrix} B(n) \end{bmatrix} \begin{bmatrix} y_n \\ \vdots \\ y_{n+s-1} \\ \boldsymbol{S_n} \end{bmatrix}$$



Numerical Analytic Continuation



$\arctan\left(\frac{5}{4}(1+i)\right)$?

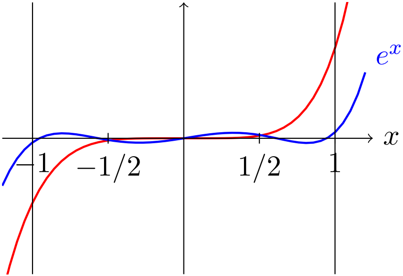
$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\ldots + 0.22\ldots \\ 0 & 0.72\ldots - 0.20\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.36\ldots + 0.32\ldots \\ 0 & 0.75\ldots - 0.07\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

...

For a fixed function y and a fixed evaluation point $z \in \mathbb{C}$,
one may compute $y(z)$ with absolute error $\leqslant 2^{-n}$
in time $\boldsymbol{O(M(n\log^2 n))}$ and space $\boldsymbol{O(n)}$. [Chudnovsky^2, v.d. Hoeven, M.]

Polynomial Approximation on Intervals



$$e^x - (1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4)$$

$$e^x - (12660T_0(x) + 1.1303T_1(x) \\ + 0.2714T_2(x) + 0.0443T_3(x) \\ + 0.0054T_4(x))$$

$$y(x) = \sum_{n \in \mathbb{Z}} c_n T_n(x)$$

$$c_{n+1} + 2\,n\,c_n - c_{n-1} = 0$$

Obstacles:

- high order ↔ divergent solutions
- ini. cond. $\notin \mathbb{Q}$

Solution: [Benoit-Joldeș-M.]

- backward recursion (“Clenshaw’s algo”)

Error Bounds

Majorizing series (a priori bounds)

- Bound the differential equation with a simple "model equation":

$$y'(z) = a(z)\, y(z) \qquad \lll \qquad g'(z) = \frac{1}{(1-\alpha\, z)}\, g(z)$$

- Solve the model equation and study the solutions:

$$\left| \sum_{k=n}^{+\infty} y_n\, z^n \right| \leqslant \sum_{k=n}^{+\infty} g_n\, |z|^n \leqslant ?$$

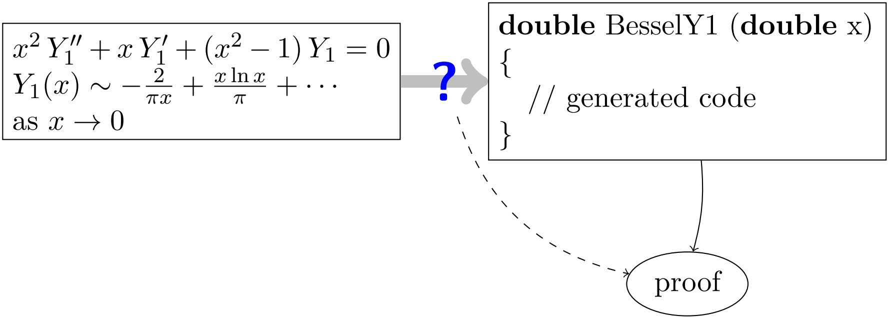
Fixed-point bounds (a posteriori bounds)

- Rewrite as $T(y) = y$, say with $\|T(f) - T(g)\| \leqslant \lambda\, \|f - g\|$, $\lambda < 1$
- Given an approximate solution $\tilde{y}$, check that $T(B(y, \varepsilon)) \subseteq B(y, \varepsilon)$

Double-Precision Implementation
The Problem

- Writing floating-point implementations of mathematical functions is hard!
- Special function libraries **performance** and **rigor** decades behind state-of-the-art elementary function libraries
- Libraries rarely provide "the right" function
  - $\exp(\sin(x))$
  - not accurate enough, more accurate than needed...

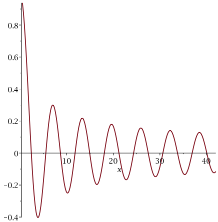
Code Generation



A Prototype Code Generator [Lauter-M. 2014]

Input: differential equation + initial values defining f
domain $[a,b]$
target accuracy
(+ implementation constraints)

Output: C code
$$\left|\frac{\texttt{implem}(x) - f(x)}{f(x)}\right| \leqslant \varepsilon \text{ for all } x \in [a,b] \cap \texttt{ double}$$



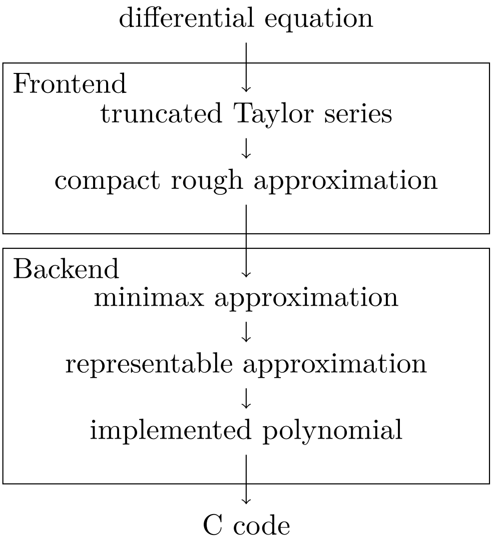
Example:

$f(x) = J_0(x)$ (Bessel)

$[a,b] = [0.5, 42]$

$\varepsilon = 2^{-45}$

Architecture



- "Rigorous **polynomial approximation** pipeline"

- **Interesting points**:
  $f(x) = 0$ or $f(x) \approx 0$
  $x = 0$, $x \to \infty$
  later: singularities

- Reduce to implementing well-chosen polynomials