

Space Complexity of Fast D-Finite Function Evaluation

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Binary Splitting

A classical method to evaluate series of rational numbers

- ▶ Classical constants

$$\text{Ex.: } \frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k+3/2}}$$

[Chudnovsky & Chudnovsky 1989]

- ▶ Elementary functions [Brent 1976, ...]

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- ▶ **D-Finite** functions

D-Finite Functions

[Stanley 1980, Zeilberger 1990, ...]

An analytic function $y(z) : \mathbb{C} \rightarrow \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear (homogeneous) ODE with polynomial coefficients:

$$a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z].$$

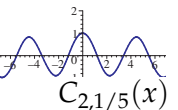
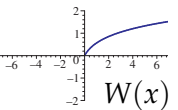
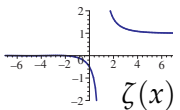
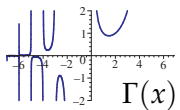
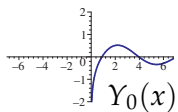
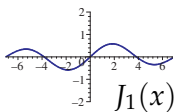
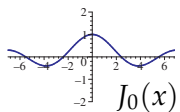
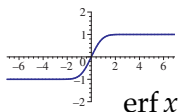
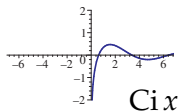
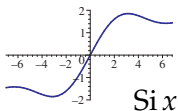
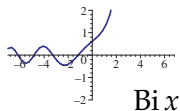
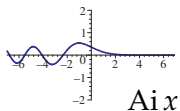
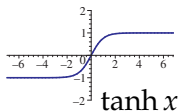
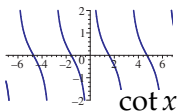
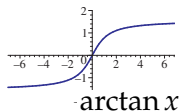
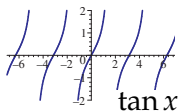
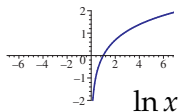
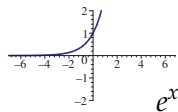
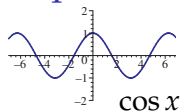
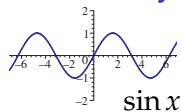
The sequence of Taylor coefficients of a D-finite functions obeys a linear **recurrence relation** with polynomial coefficients.

Example: $y(z) = \sum_{n \geq 0} y_n z^n = \exp z$

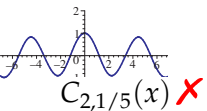
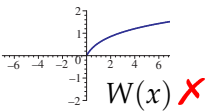
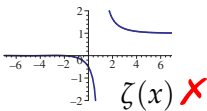
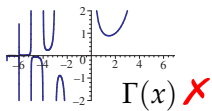
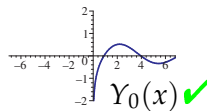
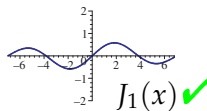
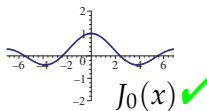
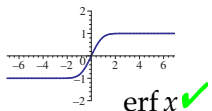
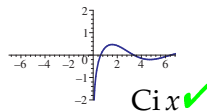
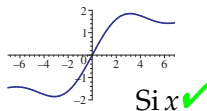
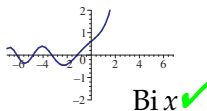
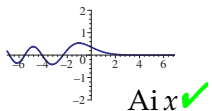
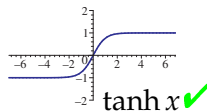
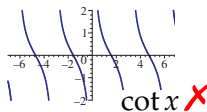
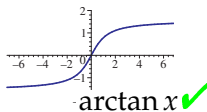
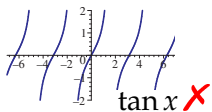
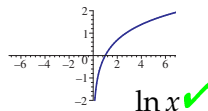
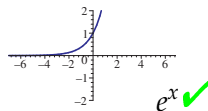
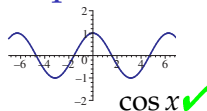
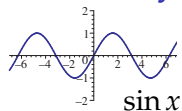
$$y'(z) - y(z) = 0 \qquad y(0) = 1$$

$$(n+1) y_{n+1} - y_n = 0 \qquad y_0 = 1$$

Elementary and Special Functions



Elementary and Special Functions



Main Result

Theorem

“D-finite functions can be evaluated in quasi-linear time and linear space.”

That is: Fix a D-finite function y and a point $\zeta \in \mathbb{C}$.

The value $y(\zeta)$ may be computed with absolute error $\leq 2^{-d}$ in $O(M(d)(\log d)^2)$ operations, using $O(d)$ bits of memory.

Here $M(n) = \text{compl. of } \leq n\text{-bit integer mult.} = O(n(\log n)e^{O(\log^* n)})$

Previous results

Same time complexity, space $O(d \log d)$

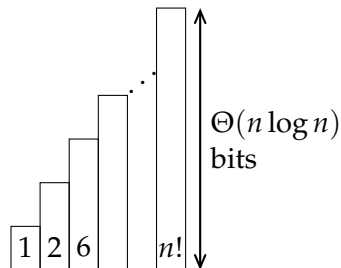
[Chudnovsky & Chudnovsky 1990, van der Hoeven 1999, M. 2010]

Space $O(d)$ in special cases [Brent 1976, ..., Yakhontov 2011]

Warm-Up: Computing $n!$

Naïve algorithm

$$n! = n \times (n - 1)!$$

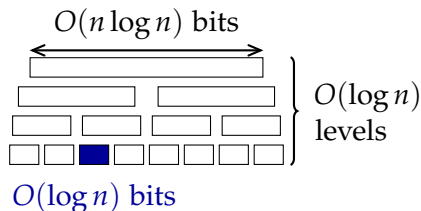


Time: $\Omega(n^2 \log n)$

Space: $O(n \log n)$

Binary splitting (= product tree)

$$n! = (\lfloor n/2 \rfloor \cdots n) \times (1 \cdots \lfloor n/2 \rfloor)$$



Time: $O(M(n \log n) \log n)$

Space: $O(n \log n)$

Standard assumption: $M(n + m) \leq M(n) + M(m)$.

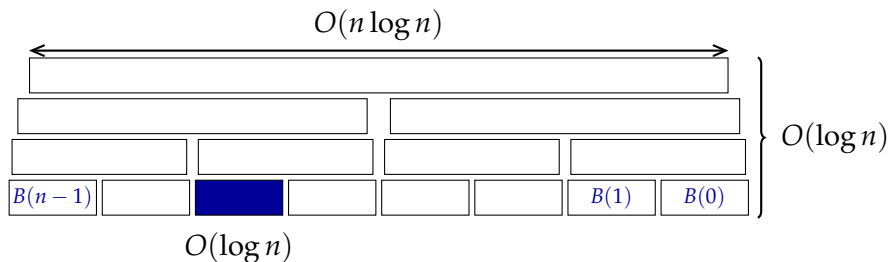
In the special case of $n!$: time $O(M(n \log n))$ [Borwein, Schönhage].

Binary Splitting for D-Finite Functions

Same idea, using a matrix product tree

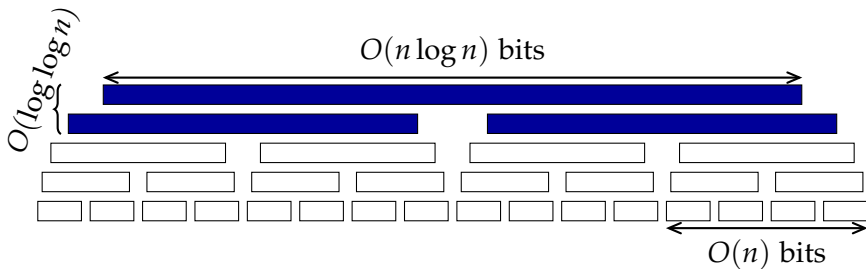
$$y(z) = \sum_{n=0}^{\infty} y_n z^n \quad Y_{n+1} = C(n)Y_n, \quad Y_n = (y_n, \dots, y_{n+s-1})^T$$

$$S_n = \sum_{k=0}^{n-1} y_k \zeta^k \quad \begin{bmatrix} Y_{n+1} \\ S_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \zeta C(n) & 0 \\ 1, 0, \dots & 1 \end{bmatrix}}_{B(n)} \begin{bmatrix} Y_n \\ S_n \end{bmatrix}$$



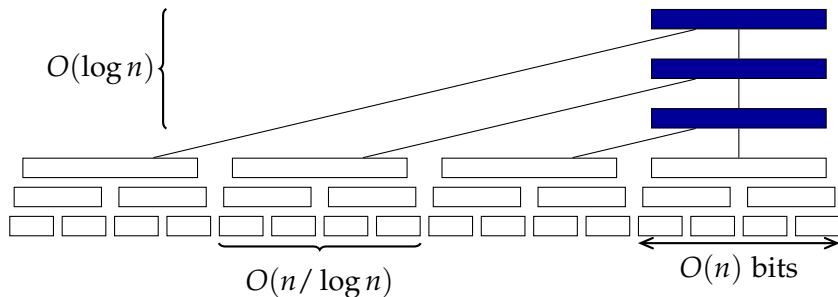
The Truncation Trick

[Gourdon & Sebah, 1996?]



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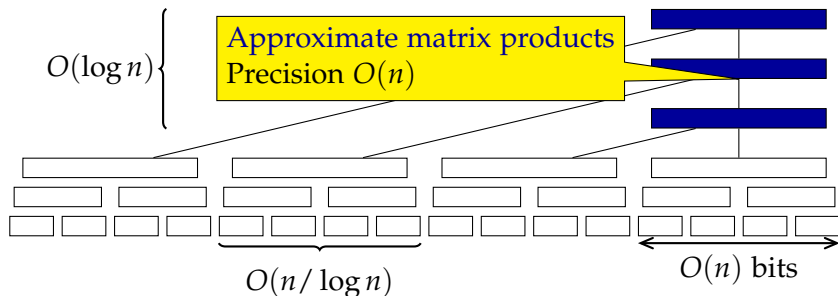


Time: $O(M(n \log n) \log n)$ (as before)

Space: $O(n)$

The Truncation Trick

[Gourdon & Sebah, 1996?]

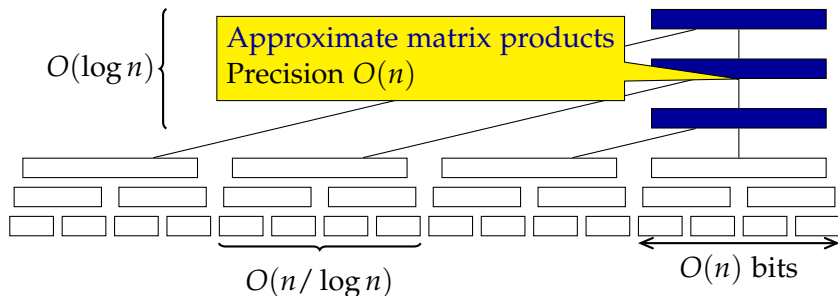


Time: $O(M(n \log n) \log n)$ (as before)

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The Truncation Trick

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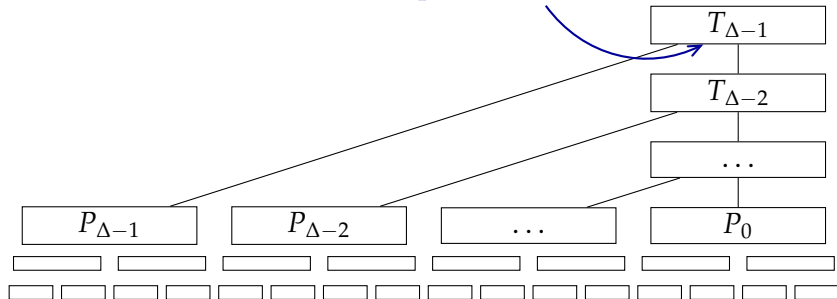
Space: $O(n)$

...provided the roundoff errors do not interfere!

[Yakhontov 2011 — Hypergeometric case]

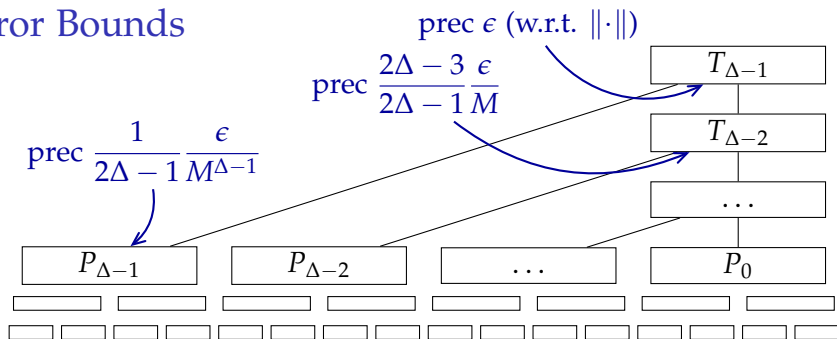
Error Bounds

prec ϵ (w.r.t. $\|\cdot\|$)



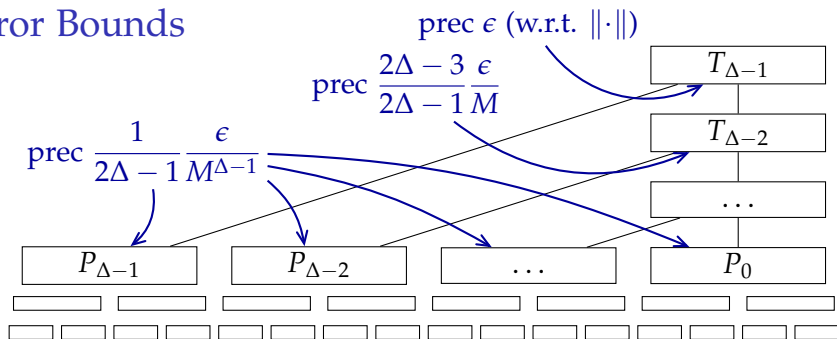
- ▶ Assume $\|P_0\|, \|P_1\|, \dots \leq M$

Error Bounds



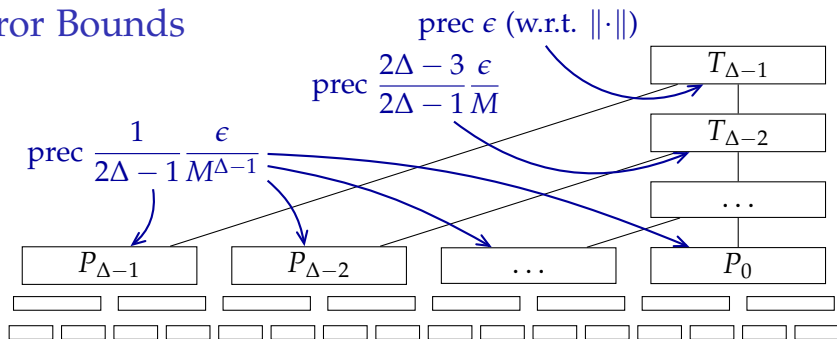
- ▶ Assume $\|P_0\|, \|P_1\|, \dots \leq M$
- ▶ $\|\tilde{Q}\tilde{P} - QP\| \leq \|\tilde{Q} - Q\| \|P\| + \|\tilde{Q}\| \|\tilde{P} - P\|$

Error Bounds



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Error Bounds



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- ▶ $\|\tilde{Q}\tilde{P} - QP\| \leq \|\tilde{Q} - Q\| \|P\| + \|\tilde{Q}\| \|\tilde{P} - P\|$
- ▶ $P_i = B(\lfloor \frac{i+1}{\Delta} n \rfloor - 1) \cdots B(\lfloor \frac{i}{\Delta} n \rfloor)$, so $M \leq C^{\lceil n/\Delta \rceil}$
- ▶ Max working prec $\approx \frac{1}{2\Delta} \frac{\epsilon}{M^\Delta} \approx \frac{1}{2\Delta} \frac{\epsilon}{C^n}$,
i.e., $\approx \log \epsilon^{-1} + n \log C + o(n) = O(n)$ digits

Prototype Implementation in NumGfun

```
> diffeq := collect({diff(y(z),z,z)-(-2*
z^5+4*z^3+z^4*a-2*z-a)*(diff(y(z),z))/(
(z-1)^3*(z+1)^3)-(-z^2*b+(-c-2*a)*z-d)*y
(z)/((z-1)^3*(z+1)^3),y(0)=1,D(y)(0)=0},
diff,factor);
```

$$\text{diffeq} := \left\{ \frac{d^2}{dz^2} y(z) - \frac{(-2z^3 + z^2 a + 2z + a) \left(\frac{d}{dz} y(z) \right)}{(z+1)^2 (z-1)^2} + \frac{(z^2 b + zc + 2za + d) y(z)}{(z-1)^3 (z+1)^3}, y(0) = 1, D(y)(0) = 0 \right\}$$

```
> a, b, c, d := 1, 1/3, 1/2, 3;
a, b, c, d := 1, 1/3, 1/2, 3
```

```
> evalf[51](HeunD(a, b, c, d, 1/3));
1.23715744756395253918007831405821000395447403052069
```

```
> myHeunD := diffeqtoproc(diffeq, y(z)):
```

```
> myHeunD(1/3, 50);
1.23715744756395253918007831405821000395447403052075
```

```
> diffeq := random_diffeq(3, 2);
```

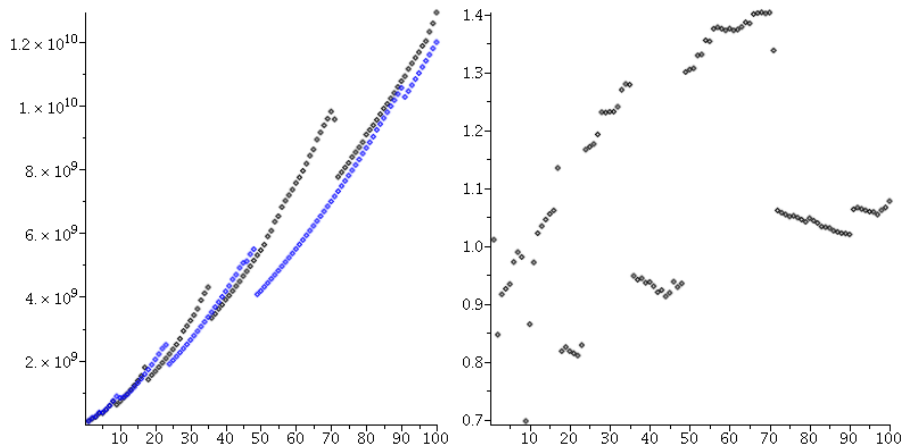
$$\text{diffeq} := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
0.0448555748776784313189330814759311548663
+ 0.0199048983021280530504789772581099788282 I
```



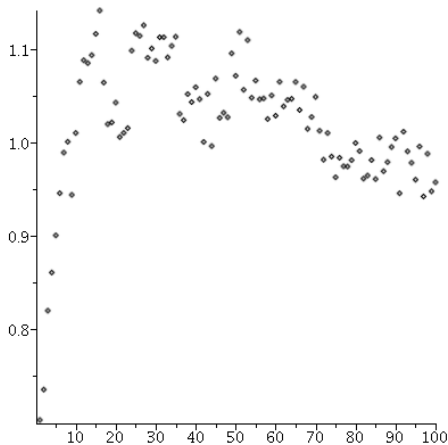
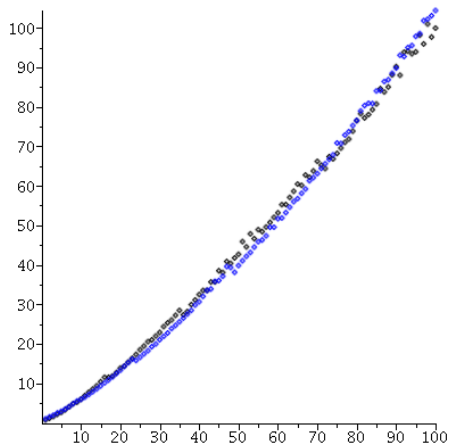
<http://algo.inria.fr/libraries/papers/gfun.html>
(GNU LGPL)

(Very Preliminary) Experimental Results: Space



Diff. eq. from previous slide, $z = 1/5$, prec = 1 000, 2 000, ..., 100 000
 Left: black = classical, blue = truncated Right : classical/truncated

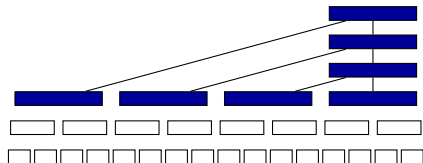
(Very Preliminary) Experimental Results: Time





Summary

- ▶ D-Finite functions may be evaluated in quasi-linear time and **linear space**
- ▶ Proof based on Chudnovsky & Chudnovsky's **binary splitting** algorithm + **effective error bounds**
- ▶ Prototype implementation available (pure Maple)



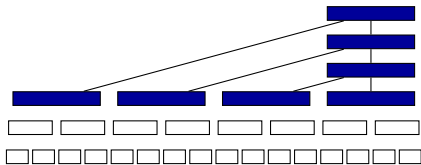
Current & future work

- ▶ Make it practical!
- ▶ Seems to require a more careful / lower-level implementation



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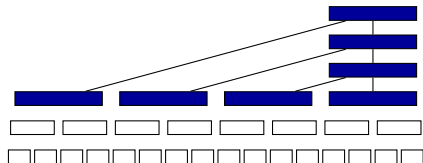
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Thank you!



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