Numerical Evaluation of D-Finite Functions
NumGfun and Beyond

Marc Mezzarobba

RISC, JKU Linz

Sage Days 49, Orsay
2013-06-20
D-Finite Functions
Elementary and Special Functions

\[
\begin{align*}
\sin x & \quad \cos x & \quad e^x & \quad \ln x \\
\tan x & \quad \arctan x & \quad \cot x & \quad \tanh x \\
\text{Ai} x & \quad \text{Bi} x & \quad \text{Si} x & \quad \text{Ci} x \\
\text{erf} x & \quad J_0(x) & \quad J_1(x) & \quad Y_0(x) \\
\Gamma(x) & \quad \zeta(x) & \quad W(x) & \quad C_{2,1/5}(x)
\end{align*}
\]
An analytic function $y(z) : \mathbb{C} \to \mathbb{C}$ is said to be D-finite (holonomic) iff it satisfies a linear (homogeneous) ODE with polynomial coefficients:

$$a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z].$$

The sequence of Taylor coefficients of a D-finite functions obeys a linear recurrence relation with polynomial coefficients.

**Example:** $y(z) = \sin z$

$$y''(z) + y(z) = 0 \quad y(0) = 0, \quad y'(0) = 1$$
D-Finite Functions

An analytic function \( y(z) : \mathbb{C} \rightarrow \mathbb{C} \) is said to be D-finite (holonomic) iff it satisfies a linear (homogeneous) ODE with polynomial coefficients:

\[
a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z].
\]

- The sequence of Taylor coefficients of a D-finite functions obeys a linear recurrence relation with polynomial coefficients.

Example: \( y(z) = K_0(z) \) (modified Bessel function)

\[
z y''(z) + y'(z) - z y(z) = 0
\]
A formal power series \( y(z) \in \mathbb{K}[[z]] \) is said to be \textbf{D-finite} (holonomic) iff it satisfies a linear (homogeneous) ODE with polynomial coefficients:

\[
a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z].
\]

- A formal power series is D-finite iff its coefficients obey a linear recurrence relation with polynomial coefficients.
- Symbolic specifications [Joyal, Flajolet...] translate into algebraic / differential equations

Example: \( y(z) = \sum_{n=0}^{\infty} n!z^n \)

\[
z^2 y''(z) + (3z - 1) y'(z) + y(z) = 0
\]
Elementary and Special Functions

\begin{align*}
\sin x & \quad & \cos x & \quad & e^x & \quad & \ln x \\
\tan x & \quad & \arctan x & \quad & \cot x & \quad & \tanh x \\
\text{Ai} x & \quad & \text{Bi} x & \quad & \text{Si} x & \quad & \text{Ci} x \\
\text{erf} x & \quad & J_0(x) & \quad & J_1(x) & \quad & Y_0(x) \\
\Gamma(x) & \quad & \zeta(x) & \quad & W(x) & \quad & C_{2,1/5}(x)
\end{align*}
Elementary and Special Functions

- $\sin x
- \cos x
- e^x
- \ln x
- \tan x
- \arctan x
- \cot x
- \tanh x
- \text{Ai } x
- \text{Bi } x
- \text{Si } x
- \text{Ci } x
- \text{erf } x
- J_0(x)
- J_1(x)
- Y_0(x)
- \Gamma(x)
- \zeta(x)
- W(x)
- C_{2,1/5}(x)

\text{ad hoc code vs uniform framework}

Marc Mezzarobba (RISC, JKU Linz) Numerical Evaluation of D-Finite Functions
A Dictionary of D-Finite Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Select a special function from the list

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project

Contents
- The inverse cosecant arccsc(x)
- The inverse cosine arccos(x)
- The inverse cotangent arccot(x)
- The inverse hyperbolic cosecant arccsch(x)
- The Airy function of the first kind Ai(x)
- The inverse secant arcsec(x)
- The inverse sine arcsin(x)
- The inverse tangent arctan(x)
- The Airy function (of the second kind) Bi(x)
- The hyperbolic cosine integral Chi(x)
- The cosine integral Ci(x)
- The cosine cos(x)
- The exponential integral Ei(x)
- The error function erf(x)
- The complementary error function erfc(x)
- The imaginary error function erfi(x)
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Contents
- The inverse cosecant \( \text{arccsc} (x) \)
- The inverse cosine \( \text{arccos} (x) \)
- The inverse cotangent \( \text{arccot} (x) \)
- The inverse hyperbolic cosecant \( \text{arccsch} (x) \)
- The Airy function of the first kind \( \text{Ai} (x) \)
- The inverse secant \( \text{arcsec} (x) \)
- The inverse sine \( \text{arcsin} (x) \)
- The inverse tangent \( \text{arctan} (x) \)
- The Airy function (of the second kind) \( \text{Bi} (x) \)
- The hyperbolic cosine integral \( \text{Chi} (x) \)
- The cosine integral \( \text{Ci} (x) \)
- The cosine \( \text{cos} (x) \)
- The exponential integral \( \text{Ei} (x) \)
- The error function \( \text{erf} (x) \)
- The complementary error function \( \text{erfc} (x) \)
- The imaginary error function \( \text{erfi} (x) \)
The Special Function $Ai(x)$

1. Differential equation

The function $Ai(x)$ satisfies

$$\frac{d^2}{dx^2} y(x) - xy(x) = 0$$

with initial values $y(0) = 1/3 \frac{\sqrt[3]{3}}{\Gamma(2/3)}, (y')(0) = -1/2 \frac{\sqrt[3]{3}\Gamma(2/3)}{\pi}$.

2. Plot of $Ai(x)$
A Dictionary of D-Finite Functions

1. Differential equation

The function $\text{Ai}(x)$ satisfies

$$\frac{d^2}{dx^2} y(x) - xy(x) = 0$$

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2. Plot of $\text{Ai}(x)$
A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai} \left( \frac{1}{4} + \frac{1}{4} i \right) \approx 0.28881085 - 0.06285935 i. \]

(Below, path may be either a point \( z \) or a broken-line path \([z_1, z_2, \ldots, z_n]\) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^* i \).)

path = \( \frac{1}{4} + \frac{1}{4} * i \) precision = 8

Submit Query

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of Airy\( \text{Ai} \) at 0:

\[ \text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{3^{3/2} x^3}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n + 1}{9^n \Gamma(n + 4/3) n!}. \]
A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai} \left( \frac{1}{4} + \frac{1}{4}i \right) \approx 0.28881085 - 0.06285935i. \]

(Below, path may be either a point \( z \) or a broken-line path \([z_1, z_2, \ldots, z_n]\) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^*i \).)

\[ \text{path} = 1/4 + 1/4i \quad \text{precision} = 8 \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of AiryAi at 0:

\[ \text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt{3} x^3 n}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n+1}{9^n \Gamma(n + 4/3) n!}. \]
A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai} \left( \frac{1}{4} + \frac{1}{4} i \right) \approx 0.28881085 - 0.06285935 i. \]

(Below, path may be either a point \( z \) or a broken-line path \( \{ z_1, z_2, \ldots, z_n \} \) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^* i \).)

\[ \text{path} = \frac{1}{4} + \frac{1}{4} i \quad \text{precision} = 80 \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of AiryAi at 0:

\[ \text{Ai}(x) = \sum_{n=0}^{\infty} \frac{\sqrt{3} x^3}{9^n \Gamma(n + 2/3) n!} - \frac{3^{2/3} x^3}{9^n \Gamma(n + 4/3) n!}. \]
3. Numerical Evaluation

\[ \text{Ai}(1/4 + 1/4 i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \]

(Below, path may be either a point \( z \) or a broken-line path \([z_1, z_2, \ldots, z_n]\) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^* i \).)

\[ \text{path} = 1/4 + 1/4^* i \quad \text{precision} = 80 \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of \( \text{AiryAi} \) at 0:

\[ \text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt{3} x^3 n}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n + 1}{9^n \Gamma(n + 4/3) n!} \]
A Dictionary of D-Finite Functions

▶ computed from the ODE
▶ rigorous error bounds
▶ arbitrary precision

3. Numerical Evaluation

\[ \text{Ai} \left( \frac{1}{4} + \frac{1}{4} i \right) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \]

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Submit Query

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of AiryAi at 0:

\[
\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^3 n}{9^n \Gamma(n+2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n+1}{9^n \Gamma(n+4/3) n!}.
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A Dictionary of D-Finite Functions

- computed from the ODE
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- arbitrary precision
A Dictionary of D-Finite Functions

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\( \text{Ai}(1/4 + 1/4 i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \)

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\[ \text{path} = \frac{1}{4} + \frac{1}{4} i \quad \text{precision} = 80 \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

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\text{Ai}(x) = \sum_{n=0}^{\infty} 1/3 \frac{\sqrt{3}x^3n}{9^n \Gamma(n + 2/3)n!} - 1/9 \frac{3^{2/3}x^3n+1}{9^n \Gamma(n + 4/3)n!}.
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A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai}(1/4 + 1/4 \, i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \]

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\[
\text{path} = 1/4 + 1/4 \, i \quad \text{precision} = 80
\]

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A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai}(1/4 + 1/4 i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \]

(Below, path may be either a point \( z \) or a broken-line path \( [z_1, z_2, \ldots, z_n] \) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^* \text{i} \).)

\[ \text{path} = -5 \quad \text{precision} = 80 \quad \text{Submit Query} \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of AiryAi at 0:

\[
\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\sqrt{3} x^3 n^{3/2}}{9^n \Gamma(n + 2/3)} - \frac{1}{9} \frac{3^{2/3} x^3 n^{1+1}}{9^n \Gamma(n + 4/3)}.
\]
A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai}(1/4 + 1/4 i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647 \]

(Below, path may be either a point \( z \) or a broken-line path \([z_1, z_2, \ldots, z_n]\) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^*i \).)

path = -5, precision = 80

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of \( \text{AiryAi} \) at 0:

\[
\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \sqrt{3} x^3 \frac{3^n n}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n + 1}{9^n \Gamma(n + 4/3) n!}.
\]
3. Numerical Evaluation

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\[ \text{path} = -5 \quad \text{precision} = 800 \]

4. Taylor expansion of \( \text{Ai}(x) \) at 0

- Expansion of \( \text{AiryAi} \) at 0:

\[
\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^3 n}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^3 n^{1+1}}{9^n \Gamma(n + 4/3) n!}.
\]
A Dictionary of D-Finite Functions

3. Numerical Evaluation

\[ \text{Ai}(-5) \approx 0.35076100902411431978801632769674221484443250893087208211128178049911192682 \]

(Below, path may be either a point \( z \) or a broken-line path \([z_1, z_2, \ldots, z_n]\) along which to perform analytic continuation of the solution of the defining differential equation. Each \( z_k \) should be of the form \( x + y^*i \).)

\[ \text{path} = -5 \quad \text{precision} = 800 \]

Submit Query

4. Taylor expansion of \( \text{Ai}(x) \) at \( 0 \)

- Expansion of AiryAi at 0:

\[
\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt{3}x^3 n}{9^n \Gamma(n + 2/3)n!} - \frac{1}{9} \frac{3^{2/3}x^3 n+1}{9^n \Gamma(n + 4/3)n!}.
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</tr>
</tbody>
</table>
NumGfun

http://algo.inria.fr/libraries/ (GNU LGPL)

http://algo.inria.fr/libraries/papers/gfun.html


arctan \( z \)

\[
(1 + z^2) y''(z) + 2z y'(z) = 0,
\]
\[
y(0) = 0, \quad y'(0) = 1
\]
Numerical Analytic Continuation

\[
\begin{align*}
\left[ y(z_1) \right] & = \left[ \frac{1}{4} (1 + \frac{5}{4} i) \right] \\
\left[ y(0) \right] & = \left[ \frac{1}{4}(1 - \frac{5}{4} i) \right]
\end{align*}
\]
Numerical Analytic Continuation

\[
\begin{bmatrix}
  y(z_1) \\
  y'(z_1)
\end{bmatrix} =
\begin{bmatrix}
  1 & 0.5705170238 \cdots + 0.2200896807 \cdots i \\
  0 & 0.7288378766 \cdots - 0.2065997130 \cdots i
\end{bmatrix}
\begin{bmatrix}
  y(0) \\
  y'(0)
\end{bmatrix}
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Numerical Analytic Continuation

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\begin{bmatrix}
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y(0) \\
y'(0)
\end{bmatrix}
\]
Numerical Analytic Continuation

\[
\begin{bmatrix}
y(z_1) \\
y'(z_1) \\
y(z_2) \\
y'(z_2)
\end{bmatrix} =
\begin{bmatrix}
1 & 0,5705170238 \cdots + 0,2200896807 \cdots i \\
0 & 0,7288378766 \cdots - 0,2065997130 \cdots i \\
1 & 0,3656231471 \cdots + 0,3290407483 \cdots i \\
0 & 0,7515011402 \cdots - 0,0792619810 \cdots i
\end{bmatrix}
\begin{bmatrix}
y(0) \\
y'(0) \\
y(z_1) \\
y'(z_1)
\end{bmatrix}
\]
Numerical Analytic Continuation

\[
\begin{bmatrix}
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y'(z_1) \\
y(z_2) \\
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1 & 0.3656231471 \cdots + 0.3290407483 \cdots i \\
0 & 0.7515011402 \cdots - 0.0792619810 \cdots i 
\end{bmatrix} \begin{bmatrix} y(0) \\
y'(0) \\
y(z_1) \\
y'(z_1) 
\end{bmatrix}
\]
Regular Singular Points

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z)$$

0 singular point

regular singular

irregular singular

for any solution $y$,
$$\exists N \text{ s.t. } y(z) = O\left(\frac{1}{|z|^N}\right)$$
as $z \to 0$

Ex.: $y(z) = z^{\sqrt{2}}, y(z) = \frac{\log z}{z}$

non-poly. growth (w.r.t. $1/|z|$) possible

as $z \to 0$

Ex.: $y(z) = e^{1/z}$

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Solutions at Regular Singular Points

**Theorem** [Fuchs, 1866]

Assume 0 is a regular singular point of an ODE with meromorphic coefficients. Then, on some neighborhood $D$ of 0, there exists a basis of solutions of the form

$$z^\lambda (y_0(z) + y_1(z) \log z + \cdots + y_t(z) \log^t z), \quad z \in D \setminus \{0\}$$

where $\lambda \in \bar{\mathbb{Q}}$ and the $y_i$ are **analytic** on $D$. 

Marc Mezzarobba (RISC, JKU Linz) Numerical Evaluation of D-Finite Functions
Asymptotics of Linear Recurrence Sequences

This is AsyRec, A Maple package accompanying Doron Zeilberger's article:

It finds the asymptotics of solutions of (homog.) linear recurrence equations with polynomial coefficients, using the Birkhoff-Trjitzinsky method.

\[ recop := (n+2)^2N^2-(7n^2+21n+16)N-8(n+1)^2; \]

\[ \text{AsyC}(recop, n, N, 5, [2, 10], 1000); \]

\[ 8^n \left( 1 - \frac{1}{3n} + \frac{1}{27n^2} + \frac{1}{81n^3} + \frac{1}{243n^4} + \frac{11}{2187n^5} \right) \]

Singularity Analysis

[Flajolet, Odlyzko]

**Principle**

asymptotic behaviour of \( y(z) = \sum_{n} y_n z^n \) at its singularities

mechanical transfer

asymptotic behaviour of \( (y_n) \) at infinity

- Constants by singularity analysis
  + numerical analytic continuation
  [Flajolet & Puech 1986]
Outlook
# D-Finite Functions in Sage

## What is there
- Nothing right now
- Arithmetic of diff. operators via PLURAL’s G-algebras

## Main goals
Modern versions of the main features of
- gfun
- Mgfun
- NumGfun
- (part of) DEtools
- ...

## A more ambitious goal
D-Finite functions as “first-class citizens”
Use them to implement special functions
(Cf. DDMF)
Developments Planned or in Progress (that I know of)

- Fredrik Johansson, Manuel Kauers, Maximilian Jaroschek
  ore_algebra 0.1 released two days ago!
  Ore operators, closure properties, guessing. . .

- ANR Magnum
  tools for analytic combinatorics project(?)

- Matthieu Dien, Marguerite Zamansky
  multivariate lazy power series prototype

- Eviatar Bach (mentored by Burcin Erocal and Flavia Stan)
  special functions, in part via D-finiteness GSOC project
Beyond NumGfun

Why Maple?

Historical reasons...
It was a pain.

Current plans

- C/C++ library
- basic analytic continuation code in arb (with Fredrik Johansson)
- Sage interface?
### Wishlist

<table>
<thead>
<tr>
<th>Feature</th>
<th>Maple</th>
<th>Sage</th>
</tr>
</thead>
<tbody>
<tr>
<td>a compiled language</td>
<td>✗</td>
<td>limited</td>
</tr>
<tr>
<td>a type system</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>sane semantics</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>differential operators, D-finite funs</td>
<td>✓</td>
<td>soon?</td>
</tr>
<tr>
<td>floating-point, interval arithmetic</td>
<td>minimal</td>
<td>✓</td>
</tr>
<tr>
<td>algebraic numbers</td>
<td>limited</td>
<td>✓ (?)</td>
</tr>
<tr>
<td>symbolic special functions, branch cuts...</td>
<td>✓</td>
<td>minimal</td>
</tr>
<tr>
<td>asymptotics</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>ability to fix/extend the system!</td>
<td>✗</td>
<td>✓</td>
</tr>
</tbody>
</table>

Marc Mezzarobba  (RISC, JKU Linz)  Numerical Evaluation of D-Finite Functions
Making Numerics Reliable

Real/complex mid-rad interval arithmetic, aka ball arithmetic

$$(3.14159265358979323846264338328, \quad 2 \cdot 10^{-30})$$

- multiple-precision floating-point number
- machine precision (rel?) error bound

- Make balls the **default** for RR, CC?
- ...in a backward-compatible way?
- Functions that do not provide guaranteed results would still be allowed to return (accurate-in-practice result, $\infty$)
d) Measure the width of the vacuum
There is a notch in the aluminum block
measure the width (width).
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0 fast integer multiplication
1 binary splitting
2 analytic continuation
3 bit burst

2. Taylor series method for ODEs

\[ \arctan\left(\frac{5}{4}(1 + i)\right) = ? \]
### Main Ideas

0. Fast integer multiplication
1. Binary splitting
2. Analytic continuation
3. Bit burst

### 2. Taylor series method for ODEs

\[
\begin{align*}
\arctan\left(\frac{5}{4}(1+i)\right) &= ? \\
\begin{bmatrix} y'(z_1) \\ y(z_1) \end{bmatrix} &= \begin{bmatrix} 1 & 0.570...+0.220...i \\ 0 & 0.728...-0.206...i \end{bmatrix} \begin{bmatrix} y'(0) \\ y(0) \end{bmatrix}
\end{align*}
\]
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0 fast integer multiplication
1 binary splitting
2 analytic continuation
3 bit burst

2. Taylor series method for ODEs

\[
\arctan\left(\frac{5}{4}(1+i)\right) = ?
\]

\[
\begin{bmatrix}
  y(z_1) \\
  y'(z_1)
\end{bmatrix} =
\begin{bmatrix}
  1 & 0.570...+0.220...i \\
  0 & 0.728...-0.206...i
\end{bmatrix}
\begin{bmatrix}
  y(0) \\
  y'(0)
\end{bmatrix}
\]
Evaluation Algorithm

Main Ideas

0 fast integer multiplication
1 binary splitting
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3 bit burst

2. Taylor series method for ODEs

\[ \text{arctan} \left( \frac{5}{4} (1 + i) \right) = ? \]

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y'(z_1)
\end{bmatrix}
= \begin{bmatrix}
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0 & 0.728... - 0.206... i
\end{bmatrix}
\begin{bmatrix}
y(0)
y'(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
y(z_2)
y'(z_2)
\end{bmatrix}
= \begin{bmatrix}
1 & 0.365... + 0.329... i \\
0 & 0.751... - 0.079... i
\end{bmatrix}
\begin{bmatrix}
y(z_1)
y'(z_1)
\end{bmatrix}
\]
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0 fast integer multiplication
1 binary splitting
2 analytic continuation
3 bit burst

2. Taylor series method for ODEs

\[ \text{arctan} \left( \frac{5}{4} (1 + i) \right) = ? \]

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\end{bmatrix}
\begin{bmatrix}
  y(z_1) \\
  y'(z_1)
\end{bmatrix}
\]

\[ \ldots \]
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0. fast integer multiplication
1. binary splitting
2. analytic continuation
3. bit burst

0. One can multiply two integers of \( \leq n \) bits in
\[ M(n) = O(n \log n 2^{O(\log^* n)}) \] bit ops [Fürer 2007].
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

- 0. fast integer multiplication
- 1. binary splitting
- 2. analytic continuation
- 3. bit burst

1. Within the disk of convergence of a Taylor expansion:
   fast series summation algorithm based on the recurrence

\[
O(N \log N)
\]

Marc Mezzarobba (RISC, JKU Linz) Numerical Evaluation of D-Finite Functions
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0  fast integer multiplication
1  binary splitting
2  analytic continuation
3  bit burst

3. High-precision inputs:
   use analytic continuation even if the series converges!

\[ z_0 = 10_2 \rightarrow z_1 = 10,1_2 \]
\[ \rightarrow z_2 = 10,101_2 \quad \sin(e) = \sin(2,718...) = ? \]
\[ \rightarrow z_3 = 10,1011011_2 \]
\[ \rightarrow z_4 = 10,101101110010100_2 \]
\[ \rightarrow \ldots \]
\[ \rightarrow z = 10.101101110010100110000\ldots\ldots_2 \approx e \]
Evaluation Algorithm

[Chudnovsky & Chudnovsky 1988]

Main Ideas

0 fast integer multiplication
1 binary splitting
2 analytic continuation
3 bit burst

Theorem

[(Chudnovsky², van der Hoeven, M.)]

The evaluation point $z$ being fixed, one may compute $y(z)$ with error bounded by $2^{-n}$ in

$$O\left(M(n \cdot (\log n)^2)\right)$$

bit operations using $O(n)$ bits of memory.
Error Bounds

\[ \sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{N-1} y_n z^n + \sum_{n=N}^{\infty} y_n z^n \]

**Compute** suitable truncation orders (and other bounds)?

A priori bounds tend to be easier to use in fast algorithms.
Bounds

van der Hoeven, M. & Salvy

Idea: Replace $y$, by a simpler function that “dominates” it.

Bound Parameters

$\kappa, \alpha, \ldots \in \mathbb{Q}$ or $\bar{\mathbb{Q}}$ s.t.

$$|y_n| \leq n!^\kappa \cdot \alpha^n \cdot \varphi(n)$$

Main Tools:
Cauchy majorants
Saddle-point method

Symbolic Bounds

- Human-Readable
  (as far as possible!)
- Asymptotically tight

Numeric Bounds

- Conservative approx.
  of parameters
- Faster (no algebraic numbers)
Credits

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