Linear ODEs
&
Divergent Series

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Disclaimer

I am not an expert

...just trying to learn about this

Werner Balser

Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations

Marius van der Put & Michael F. Singer

Galois Theory of Linear Differential Equations

Chap. 7: "Exact Asymptotics"

Singularities of Linear ODEs

$$a_r(z)\,y^{(r)}(z)+\cdots+a_1(z)y'(z)+a_0(z)\,y(z)=0,\quad a_k\in\mathbb{C}[z]$$

$\boldsymbol{a_r(0)\neq 0}$ — Ordinary point

$$y(z)=\sum_{n=0}^{\infty} y_n\,z^n$$

analytic on $D(0,\rho)$

$\boldsymbol{a_r(0)=0}$ — Singular point

$$y_i(z)=\sum_j (\log z)^j\,z^{\lambda_{i,j}}\,y_{i,j}(z)$$

$y_{i,j}$ analytic on $D(0,\rho)\setminus\{0\}$

- All $y_{i,j}(z)$ have a **pole** at 0:
  regular singular point
- Some $y_{i,j}(z)$ has an **essential singularity**:
  irregular singular point

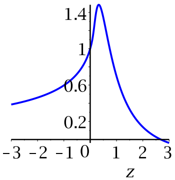
Formal Solutions

$$y_i(z) = \underbrace{e^{\mathrm{Poly}_i\left(\frac{1}{z^{1/q}}\right)} z^{\lambda_i}}_{\text{analytic in suitable domains}} \sum_j^{\text{(finite)}} (\log z)^j \underbrace{\sum_{n=0}^{\infty} c_{i,j,n}\, z^{n/q}}_{\in \mathbb{C}[[z^{1/q}]]},$$

in general divergent

Example

$$\hat{y}(z) = \sum_{n=0}^{\infty} n!\, z^n \quad \text{satisfies} \quad z^2\, \hat{y}''(z) + (3\,z - 1)\, \hat{y}'(z) + \hat{y}(z) = 0$$



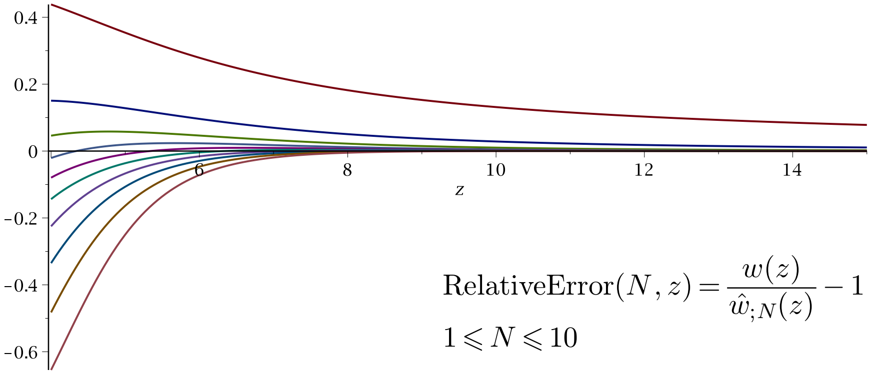
"True" solutions on $\mathbb{C} \backslash e^{i\theta}\, \mathbb{R}_+$:

$$\left\langle \frac{1}{z}\, e^{-1/z}, \quad \frac{1}{z}\, e^{-1/z}\, \mathrm{Ei}\left(\frac{1}{z}\right) \right\rangle$$

Approximation by Divergent Series

$$w(z) = z\,e^{-z}\,\mathrm{Ei}(z) \qquad w(10) \approx 1.13147$$

$$\hat{w}_{;N}(z) = \sum_{n=0}^{N-1} \frac{n!}{z^n} \qquad \hat{w}_{;10}(10) \approx 1.13159$$



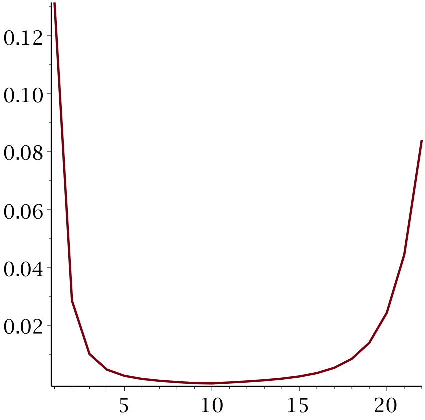
$$\mathrm{RelativeError}(N, z) = \frac{w(z)}{\hat{w}_{;N}(z)} - 1$$

$$1 \leqslant N \leqslant 10$$

Summation to the Least Term

N	$\hat{w}_{;N}(10)$
1	1.00000
2	1.10000
3	1.12000
4	1.12600
5	1.12840
6	1.12960
7	1.13032
8	1.13082
9	1.13122
10	**1.13158**
11	1.13194
12	1.13234
13	1.13282
14	1.13344
15	1.13431

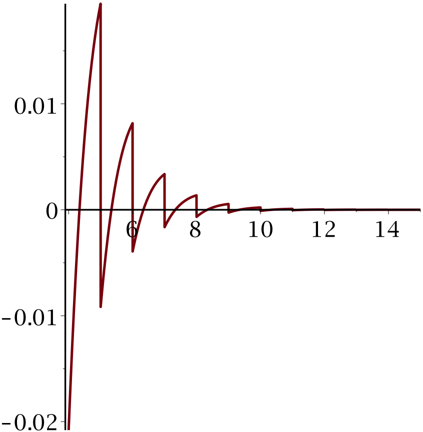
$$\boldsymbol{w(10) \approx 1.13147}$$



RelativeError$(N, 10)$

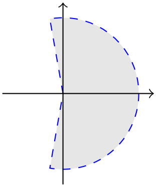
Summation to the Least Term

$$\hat{w}(10) = 1 + \frac{1}{10} + \frac{2}{10^2} + \cdots + \frac{8!}{10^8} + \mathbf{\frac{9!}{10^9}} + \mathbf{\frac{10!}{10^{10}}} + \frac{11!}{10^{11}} + \frac{12!}{10^{12}} + \cdots$$



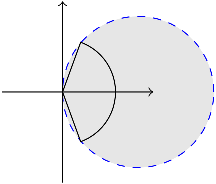
Sectors and Regions

Sector

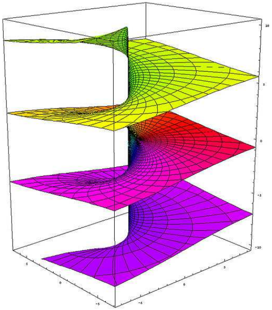


$$\left\{ z \in \dot{\mathbb{C}} : 0 < |z| < \rho_0 \atop \theta_0 < \arg z < \theta_1 \right\}$$

Sectorial region of opening α



- contained in a sector of opening α
- contains sectors of opening $\alpha - \varepsilon$



$$\dot{\mathbb{C}}$$

$$0 < \rho < \infty$$
$$\theta \in \mathbb{R}$$

(Poincaré) Asymptotic Expansions

Given y analytic in G, we write

$$y(z) \cong \hat{y}(z) = \sum_{n=0}^{\infty} y_n z^n \in \mathbb{C}[[z]] \qquad \text{as } z \to 0 \text{ with } z \in G$$

if **for all N** and for any closed sector $\Delta \subset G$,

$$y(z) = y_0 + y_1 z + \cdots + y_{N-1} z^{N-1} + O(z^N), \quad z \to 0, z \in \Delta$$

where the constant hidden in the $O(\cdot)$ may depend on N and Δ.

- A given y has at most one asymptotic expansion (its Taylor expansion!)
- Any $\hat{y}$ is the asymptotic expansion of infinitely many functions

$$e^{-1/z} = 0 + 0\,z + \cdots + 0\,z^{N-1} + O(z^N) \qquad \text{as } z \to 0,\ \operatorname{Re} z > \varepsilon$$

The Main Asymptotic Existence Theorem

Theorem [Poincaré, Horn, Birkhoff...]

In any sufficiently small sector Δ with apex at 0, there exists a basis of **analytic solutions** $y_i : \Delta \to \mathbb{C}$ with **asymptotic expansions**

$$y_i(z) \cong e^{\mathrm{Poly}_i\left(\frac{1}{z^{1/q}}\right)} z^{\lambda_i} \sum_{j}^{\text{(finite)}} (\log z)^j \sum_{n=0}^{\infty} c_{i,j,n}\, z^{n/q}, \quad z \to 0, \quad z \in \Delta.$$

Wolfgang Wasow

Asymptotic Expansions for Ordinary Difference Equations

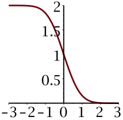
- Uniqueness?
- Computation?
- Sectors of validity?
- Link between different asymptotic lifts?

The Stokes Phenomenon

$$y''(z) + 2\,z\,y'(z) = 0$$

Analytic solutions:

$$\langle 1, \operatorname{erfc} z \rangle$$

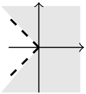


$$-\zeta^3\,w''(\zeta) + 2\,(\zeta^2 - 1)\,w'(\zeta) = 0$$

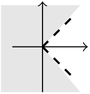
Formal solutions at ∞:

$$\left\langle 1, \quad e^{-z^2}\left(\frac{1}{z} - \frac{1}{2\,z^3} + \frac{3}{4\,z^5} - \cdots\right)\right\rangle$$

$$\operatorname{erfc}(z) \cong \frac{e^{-z^2}}{\sqrt{\pi}}\left(\frac{1}{z} - \frac{1}{2\,z^3} + \frac{3}{4\,z^5} - \cdots\right), \qquad z \to \infty \quad \text{in}$$



$$\operatorname{erfc}(z) \cong 2 + \frac{e^{-z^2}}{\sqrt{\pi}}\left(\frac{1}{z} - \frac{1}{2\,z^3} + \frac{3}{4\,z^5} - \cdots\right), \qquad z \to \infty \quad \text{in}$$



The Stokes Phenomenon

$$z - \frac{1}{2}\,z^3 + \frac{3}{4}\,z^5 - \frac{15}{8}\,z^7 + \frac{105}{16}\,z^9 - \frac{945}{32}\,z^{10} + \cdots +$$

$$\approx \sqrt{\pi}\,\exp\left(\frac{1}{z^2}\right)\left(\boldsymbol{A}\,\mathrm{erfc}\,\frac{1}{z} + \boldsymbol{B}\,1\right)$$

Gevrey Series

Formal power series of Gevrey order $s > 0$:

$$\mathbb{C}[[\boldsymbol{z}]]_{\boldsymbol{s}} = \left\{ \sum_{n \geqslant 0} y_n \, z^n \quad : \quad y_n = O\big(\boldsymbol{n!^s} \, e^{O(n)} \big) \right\}$$

(= Taylor expansions of $\mathcal{C}^\infty$ functions whose derivatives "do not grow too fast")

$$\mathbb{C}\{z\} \quad = \quad \mathbb{C}[[z]]_0 \quad \subseteq \quad \mathbb{C}[[z]]_s \quad \underset{s<s'}{\subsetneq} \quad \mathbb{C}[[z]]_{s'} \quad \subsetneq \quad \mathbb{C}[[z]]$$

Gevrey Asymptotics

Given y analytic on G, we write

$$y(z) \cong_s \hat{y}(z) = \sum_{n=0}^{\infty} y_n z^n \in \mathbb{C}[[z]] \qquad \text{as } z \to 0 \text{ with } z \in G$$

if for all N and for any closed sector $\Delta \subset G$,

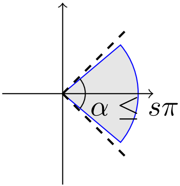
$$|y(z) - (y_0 + y_1 z + \cdots + y_{N-1} z^{N-1})| \leqslant \boldsymbol{c\, K^N N!^s}$$

where $\boldsymbol{c, K}$ **do not depend on** $\boldsymbol{N}$ (but may depend on Δ).

$$y = \hat{y} \in \mathbb{C}\{z\} \quad \Leftrightarrow \quad y \cong_0 \hat{y} \quad \Rightarrow \quad y \cong_s \hat{y} \quad \underset{s < s'}{\Rightarrow} \quad y \cong_{s'} \hat{y} \quad \Rightarrow \quad y \cong \hat{y}$$

Uniqueness

Narrow Regions



Proposition ("Ritt's Theorem")

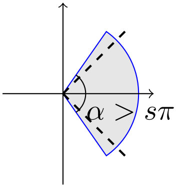
Let $G \subset \dot{\mathbb{C}}$ of opening $\leqslant s\,\pi$, and let $\hat{f} \in \mathbb{C}[[z]]_s$.

Then there exists an analytic function $f\colon G \to \mathbb{C}$ s.t.

$$f \cong_s \hat{f} \quad \text{in } G.$$

Such a function is never unique.

Wide Regions



Proposition (Watson's Lemma)

Let $G \subset \dot{\mathbb{C}}$ of opening $> s\,\pi$, and let $f\colon G \to \mathbb{C}$ be an analytic fun s.t.

$$f \cong \hat{0} \quad \text{in } G.$$

Then $f \equiv 0$.

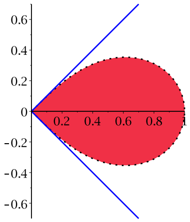
The mapping $y \mapsto$ its asymptotic expansion is never surjective.

The Laplace Transform

$$\mathcal{L}_k f(z) = z^{-k} \int_0^\infty f(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k)$$

- f analytic around $(0, \infty)$, continuous at 0
- $f(u) = O(e^{cu^k})$ as $u \to \infty$, $u \in \mathbb{R}$

- $\mathcal{L}_k f$ is defined (at least) for $\mathrm{Re}(z^{-k}) > c$
  i.e., in a region of opening π/k

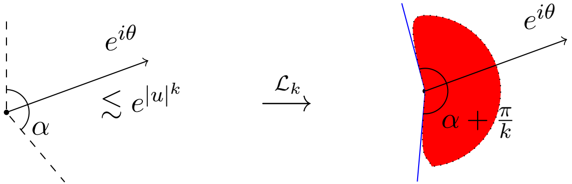


$\mathrm{Re}(z^{-2}) > 1$

The Laplace Transform in a Sector

$$\mathcal{L}_k f(z) = z^{-k} \int_0^{\boldsymbol{e^{i\theta}}\infty} f(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k)$$

- Replace $(0, \infty)$ by $(0, e^{i\theta}\infty)$
  ➡ Laplace transform on a sector



The Laplace Transform of a Monomial

$$\mathcal{L}_k f(z) = z^{-k} \int_0^\infty f(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k)$$

Laplace transform of a monomial (assume $z > 0$)

$$\begin{aligned}
\mathcal{L}_k\,(u^n)(z) &= z^{-k} \int_0^\infty u^n\, e^{-(u/z)^k}\, \mathrm{d}(u^k) \\
&= \int_0^\infty z^n\, v^{n/k}\, e^{-v}\, \mathrm{d}v && v = \left(\frac{u}{z}\right)^k \\
&= \Gamma(1 + \tfrac{n}{k})\, z^n && \mathrm{d}(u^k) = z^k\, \mathrm{d}v
\end{aligned}$$

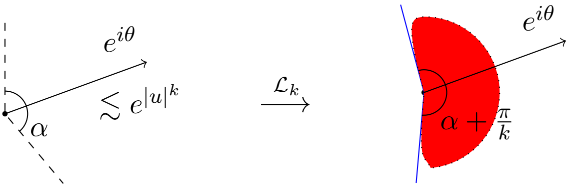
The Formal Laplace Transform

$$\hat{\mathcal{L}}_k \hat{f}(z) := \sum_n \Gamma\Big(1 + \frac{n}{k}\Big) f_n z^n, \qquad \hat{f} = \sum_n f_n u^n \in \mathbb{C}[[u]]$$

Property

In the setting of the previous slides

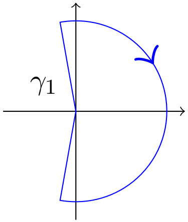
$$f \cong_s \hat{f} \quad \Longrightarrow \quad \mathcal{L}_k f \cong_{s+\frac{1}{k}} \hat{\mathcal{L}}_k \hat{f}$$



The Borel Transform

$$\mathcal{B}_k f(u) = \frac{1}{2\pi i}\int_\gamma z^k f(z)\, e^{\left(\frac{u}{z}\right)^k} \mathrm{d}(z^{-k})$$

$$\hat{\mathcal{B}}_k \hat{f}(u) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+\frac{n}{k})} u^n$$



- f analytic on a region G of opening $\alpha > \pi/k$, bounded as $z \to 0$
- $\mathcal{B}_k f$ well-defined when $\mathrm{Re}\,((u/z)^k) < 0$ on the radial rays, i.e., $|\arg u| < \varepsilon$
- Rotate the integration path to define $\mathcal{B}_k f$ on a sector of opening $\alpha - \dfrac{\pi}{k}$
- $\mathcal{B}_k f(u) \lesssim e^{|u|^k}$ as $u \to \infty$ in that region
- Respects Gevrey asymptotics

Borel-Laplace for Convergent Series

- The (analytic) Borel and Laplace transforms of order k are inverse of each other

- In particular, for convergent series :

$$f(z) = \sum_n f_n\, z^n \in \mathbb{C}\{z\} \qquad f(z) = \int_0^{\infty(d)} g(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k)$$

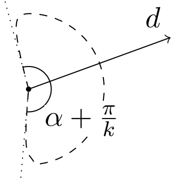
$$\downarrow \mathcal{B}_k \qquad \uparrow \mathcal{L}_k$$

$$g(u) = z^{-k} \int_0^{\infty(d)} f(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k) = \sum_n \frac{f_n}{\Gamma\left(1+\frac{n}{k}\right)}\, u^n$$

- entire function
- $\lesssim e^{-|u|^k}$

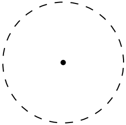
k-Summability

$$\hat f(z) = \sum_n f_n z^n \in \mathbb{C}[[z]]_{1/k} \qquad \cong_{1/k} \qquad f(z) = \int_0^{\infty(d)} g(u)\, e^{-(u/z)^k}\, \mathrm{d}(u^k)$$



$\downarrow \hat{\mathcal{B}}_k$ $\qquad\qquad\qquad\qquad$ $\uparrow \mathcal{L}_k$

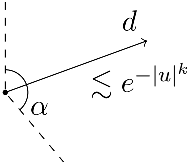
$$\hat g(u) = \sum_n \frac{f_n}{\Gamma(1+\frac{n}{k})} u^n \in \mathbb{C}\{u\} \qquad g(u)$$



analytic continuation

$\rightarrow$

"succeeds" **iff**
$\exists f,\ \hat f \cong_{1/k} f$
in a wide sector
bisected by d



Example

$$\hat{f}(z) = \sum_n n!\, z^n \in \mathbb{C}[[z]]_1 \qquad\qquad f(z) = z^{-1} \int_0^{\infty(d)} \frac{e^{-u/z}}{1-u}\, \mathrm{d}u$$

$$\downarrow \hat{\mathcal{B}}_k \qquad\qquad \uparrow \mathcal{L}_k$$

$$\hat{g}(u) = \sum_n u^n \in \mathbb{C}\{u\} \qquad \underset{u \notin \mathbb{R}_+}{\overset{\text{an. cont.}}{\rightarrow}} \qquad g(u) = \frac{1}{1-u} \ll e^{|u|}$$

E.g., with $d = e^{i \cdot \pi}$ and $z < 0$:

$$z^{-1} \int_0^{-\infty} \frac{e^{-u/z}}{1-u}\, \mathrm{d}u = -z^{-1} \int_0^{-\infty} \frac{e^v}{1 + z\, v}\, z\, \mathrm{d}v = z^{-1}\, e^{-1/z} \int_{-\infty}^0 \frac{\exp(v + z^{-1})}{v + z^{-1}}\, \mathrm{d}v$$

$$= z^{-1}\, e^{-1/z} \int_{-\infty}^{-1/z} \frac{e^{-w}}{w}\, \mathrm{d}w = z^{-1}\, e^{-1/z}\, \mathrm{Ei}(-1/z).$$

Properties of the k-Summation Operator

$\mathbb{C}\{z\}_{k,d} \subsetneq \mathbb{C}[[z]]_{1/k}$: space of k-summable functions in the direction d

$\mathcal{S}_{k,d}$: k-summation operator

$$\mathcal{S}_{k,d}(\alpha \hat{f}) = \alpha \mathcal{S}_{k,d}(\hat{f}) \qquad \mathcal{S}_{k,d}(\hat{f}\hat{g}) = \mathcal{S}_{k,d}(\hat{f})\,\mathcal{S}_{k,d}(\hat{g})$$

$$\mathcal{S}_{k,d}(\hat{f} + \hat{g}) = \mathcal{S}_{k,d}(\hat{f}) + \mathcal{S}_{k,d}(\hat{g}) \qquad \mathcal{S}_{k,d}(\hat{f}\,') = \mathcal{S}_{k,d}(\hat{f})'$$

Hence:

$$\underbrace{L\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right)}_{\substack{\text{Linear differential operator} \\ \in \mathbb{C}(x)\langle \frac{\mathrm{d}}{\mathrm{d}z} \rangle}} \cdot \overbrace{\hat{y}}^{\in \mathbb{C}\{z\}_{k,d}} = 0 \quad \Rightarrow \quad L\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right) \cdot (\mathcal{S}_{k,d}\,\hat{y}) = 0$$