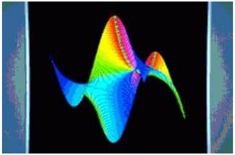
Guaranteed Precision Evaluation of D-finite Functions

Marc Mezzarobba

ALGORITHMS project, INRIA



UWO, September 12, 2008

NumGfun

- A Maple package for symbolic-numeric computation with D-finite functions and sequences in one variable
  - Guaranteed precision evaluation
  - Bounds for sequences
  - …
- Version 0.2 available (still experimental!), LGPL

  http://www.marc.mezzarobba.net/code/NumGfun-current.tgz
- Integration into gfun / algolib in progress

  http://algo.inria.fr/libraries/

Bruno Salvy and Paul Zimmermann, Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable, 1994.

Bound Computations

- Baxter permutations
  - $(n+2)(n+3)B_n = (7n^2+7n-2)B_{n-1} + 8(n-1)(n-2)B_{n-2}$, $B_0 = B_1 = 1$
  - $B_n \leq (n+8)^8 8^n$

- $t_k = \dfrac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k}}$
  - $\dfrac{12}{640320^{3/2}} \sum_{k=0}^{\infty} t_k = \dfrac{1}{\pi}$ (Chudnovsky[2] 1988)
  - $\left| \dfrac{640320^{3/2}}{12\pi} - \sum_{k=0}^{n-1} t_k \right| \leq (0.1n^4 + 0.5n^3 + 1.5n^2 + 2.1n + 1)\alpha^n$

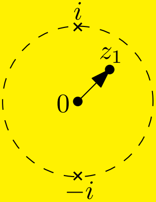
    where $\alpha = \dfrac{1}{151931373056000} \simeq 0,66 \cdot 10^{-14}$

Function Evaluation

A Familiar Example

$$(1+z^2)\ \arctan''(z) + 2z\ \arctan'(z) = 0$$

$$\arctan \frac{3(1+i)}{5} \simeq 0{,}670782196758950644190815337$$
4705632571369265547562721682009119775363456
2788546268206648547182112134208947460355580
1433079787592299964529081793221227836458496
7241027751816658681028242709786087804231203
5059588657436137542728611075919334091735855
$+\ 0{,}4313775209217135982596553539683059915248$
7122502784763704416333662458132714904677846
9188664848592351371193308077157250027646988
5281752378714171283456698686337133570545945
8746821430812351884522098343403327937148536
338890142864171080500321 i



Numerical Evaluation of Special Functions

Goal

Compute special functions to high precision $d \to \infty$

Assume $y(z) = \sum_{n=0}^{\infty} y_n z^n$.
To compute $y(z_1)$ to a (user-chosen) accuracy $\epsilon = 10^{-d}$:

1. Compute N such that $\left| y(z_1) - \sum_{n=0}^{N-1} y_n z_1^n \right| \leq \frac{\epsilon}{2}$
   $\longrightarrow$ BOUNDS
   - Van der Hoeven 1999, 2001, 2003, 2006
   - Previous slide: work in progress with B. Salvy
2. Compute $\sum_{n=0}^{N-1} y_n z_1^n$

J. van der Hoeven. Fast evaluation of holonomic functions. 1999.

J. van der Hoeven. Majorants for formal power series. 2003.

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   $\longrightarrow$ BOUNDS
2. Compute $\sum_{n=0}^{N-1} y_n z_1^n$
   - This talk

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Algorithms

"If y is D-finite, this strategy (sum the Taylor series) is competitive"

Binary splitting (Chudnovsky^2 1988):
a family of algorithms that are

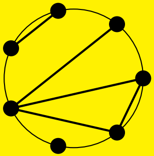
- General: whole class of D-finite functions
- Efficient: quasi-linear time complexity w.r.t. size of output
- Practical
- Actually used... in special cases only!
  (NumGfun = first general implementation?)

D.V. and G.V. Chudnovsky. Approximations and complex multiplication according to Ramanujan. 1988.

Recurrence Unrolling

An Example from Combinatorics

Motzkin Numbers



$$(n+3)\,M_{n+2} = 3n\,M_n + (2n+3)\,M_{n+1},$$
$$M_0 = 0, M_1 = M_2 = 1$$

0, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, ...

$M_{1\,000\,000}$ = 87836485521410228205552857212867952
60648460114018772686310027332206011651992742068
95017531901406553089345501470120232183076893776
76219223691237769669136651142176793088580998640
24791593930900669539159753966399354360360024084
835778 … 67840785185707760882612226992209195254
47686028065587057458044089305949409321050999808
07630126450209921669113886642195497473724754511
36778954497167179899377064889762395818323064327
4956942565741376149791829585290393680786291940

(477 112 digits)

An Example of Convergent Series

One Million Decimal Digits of π

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}} \qquad \text{(Chudnovsky}^2 \text{ 1989)}$$

$\pi \simeq$ 3,141592653589793 238462643383279502 884197169399375105 820974944592307 816406286208998628 03482534211706798 214808651328230664 709384460955058 2231725359408128 48111745028410270 …

…

879 613 033 1164 6283 9963 46460 42209010610577945815 1

Polynomially Recursive Sequences

Definition

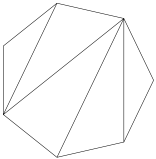
A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be P-recursive, or holonomic, if it satisfies a linear (homogenous) recurrence relation with polynomial coefficients:

$$a_s(n)\,u_{n+s} + \cdots + a_1(n)\,u_{n+1} + a_0(n)\,u_n = 0, \qquad a_j \in \mathbb{Q}(i)[n].$$

The previous sequences are P-recursive.

More Examples

- Catalan Numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$
  - Count Dyck words of length $2n$, triangulations of the convex n-gon...
  - $(n+2)C_{n+1} = (4n+2)C_n$, $C_0 = 1$



- Computing $\Gamma(z)$ for $z \in \mathbb{Q}[i]$
  - Wlog take $1 \le \operatorname{Re} z \le 2$
  - $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$$
    $$= k^z e^{-k} \sum_{n=0}^{\infty} \frac{1}{z^{\uparrow(n+1)}} k^n + \int_k^\infty e^{-t} t^{z-1} \mathrm{d}t$$
    the partial sums are P-recursive
  - Use bounds on the integral and the rest of the series to conclude

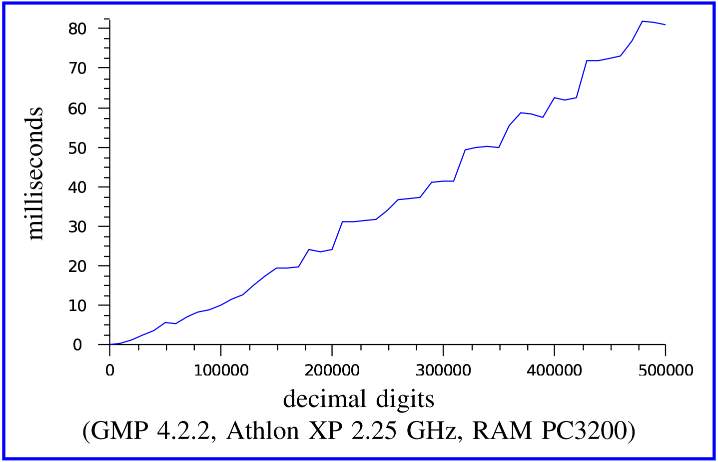
Fast Integer Multiplication

A Quick Review

- Complexity of n-digits by n-digits integer multiplication
  - naive: $\mathrm{M}(n) = \Theta(n^2)$
  - Karatsuba (1963): $\mathrm{M}(n) = \Theta(n^{\log_2 3}) = O(n^{1.59})$
  - Schönhage-Strassen (1971): $\mathrm{M}(n) = O(n \log n \log \log n)$
  - Fürer (2007): $\mathrm{M}(n) = n\,(\log n)\,2^{O(\log^* n)}$
- Fast algorithms are relevant in practice (GMP, Magma...)

Fast Integer Multiplication

A Quick Review



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  - Fürer (2007): $\mathrm{M}(n) = n\,(\log n)\,2^{O(\log^* n)}$
- Fast algorithms are relevant in practice (GMP, Magma...)
- Reduce other operations to $O(\log n)$ or even $O(1)$ multiplications
  - Division: $O(\mathrm{M}(n))$ (using Newton's method)
  - Gcd: $O(\mathrm{M}(n) \log n)$
    (“that's a lot” $\longrightarrow$ avoid gcd computations!)

Matrix Form of Recurrences

- $a_s(n)\, u_{n+s} + \cdots + a_1(n)\, u_{n+1} + a_0(n)\, u_n = 0$
- $$\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+s-1} \\ u_{n+s} \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \square & \square & \dots & \square \end{bmatrix} \begin{bmatrix} u_n \\ \vdots \\ u_{n+s-2} \\ u_{n+s-1} \end{bmatrix}$$

rational functions of n

Matrix Form of Recurrences

- $a_s(n)\, u_{n+s} + \cdots + a_1(n)\, u_{n+1} + a_0(n)\, u_n = 0$
- $$\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+s-1} \\ u_{n+s} \end{bmatrix} = \frac{1}{q(n)} \underbrace{\begin{bmatrix} q & & & \\ & \ddots & & \\ & & & q \\ \square & \square & \dots & \square \end{bmatrix}}_{A(n)} \begin{bmatrix} u_n \\ \vdots \\ u_{n+s-2} \\ u_{n+s-1} \end{bmatrix}$$

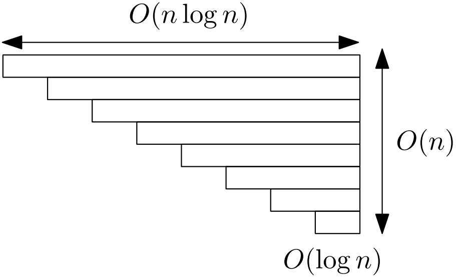
polynomials in n

Matrix Form of Recurrences

- $a_s(n)\,u_{n+s} + \cdots + a_1(n)\,u_{n+1} + a_0(n)\,u_n = 0$
- $$\begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+s-1} \\ u_{n+s} \end{bmatrix} = \frac{1}{q(n)} \underbrace{\begin{bmatrix} q & & & \\ & \ddots & & \\ & & & q \\ \square & \square & \dots & \square \end{bmatrix}}_{A(n)} \begin{bmatrix} u_n \\ \vdots \\ u_{n+s-2} \\ u_{n+s-1} \end{bmatrix}$$
- $$\begin{bmatrix} u_N \\ \vdots \\ u_{N+s-1} \end{bmatrix} = \frac{A(N-1)\cdots A(0)}{q(N-1)\cdots q(0)} \begin{bmatrix} u_0 \\ \vdots \\ u_{s-1} \end{bmatrix}$$ “Matrix factorial”

Binary Splitting

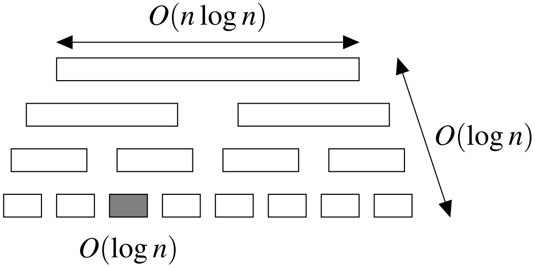
$$A(n-1)\cdots A(1)\cdot A(0)$$



Naive product:
$\Omega(n^2 \log n)$

Binary Splitting

$$A(n-1)\cdots A(1)\cdot A(0) = \left(A(n-1)\cdots A\left(\left\lfloor \tfrac{n}{2} \right\rfloor + 1\right)\right) \cdot \left(A\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right)\cdots A(0)\right)$$



$$\begin{aligned} & s^3\,\mathrm{M}(n \log n) \\ &+ 2\, s^3\,\mathrm{M}\left(\tfrac{n}{2} \log n\right) \\ &+ \cdots \\ &+ n\, s^3\,\mathrm{M}(\log n) \\ &= O\big(\mathrm{M}(n \log n) \log n\big) \end{aligned}$$

Numerical Evaluation of D-finite Functions

Elementary and Special Functions

A Familiar Example

$$(1+z^2)\ \arctan''(z) + 2z\ \arctan'(z) = 0$$

$$\arctan\frac{3(1+i)}{5} \simeq 0{,}670782196758950644190815337$$
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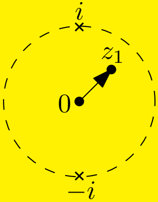
$$8746821430812351884522098343403327937148536$$
$$338890142864171080500321\ i$$



Differentially Finite Functions

Definition

A function $y(z) : \mathbb{C} \to \mathbb{C}$ is said to be D-finite (or holonomic) if it is solution to an (homogenous) linear differential equation with polynomial coefficients:

$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0, \qquad a_j \in \mathbb{Q}(i)[z].$$

Examples:

- Elementary and special functions: $\arctan(z)$, $\cos(z)$, $\mathrm{Ai}(z)$, $\mathrm{erf}(z)$, algebraic functions, hypergeometric functions...
- More general D-finite function arise in combinatorics, analysis of algorithms and number theory

D-finite Functions, P-recursive Sequences

Why Are They Interesting?

$$y(z) = \arctan(z) \quad \leftrightarrow \quad \begin{array}{l}(1+z^2)\,y''(z) + 2z\,y'(z) = 0\\ y(0) = 0,\; y'(0) = 1\end{array}$$

Some properties:

- An analytic function is D-finite iff the sequence of its Taylor coefficients is P-recursive
- Sums and products of P-recursive sequences are P-recursive
- Sums, products, derivatives, and antiderivatives of D-finite functions are D-finite

D-finite Functions, P-recursive Sequences

Why Are They Interesting?

$$\arctan(z) = \left\{ \begin{array}{l} (1+z^2)\,y''(z) + 2z\,y'(z) = 0 \\ y(0) = 0,\; y'(0) = 1 \end{array} \right\}$$

Motto

Differential Equation + Initial Values = Data Structure
(Recurrence Relation)

Some properties:

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- Sums and products of P-recursive sequences are P-recursive
- Sums, products, derivatives, and antiderivatives of D-finite functions are D-finite

Solution Space & Radius of Convergence

Cauchy's Existence Theorem for LODE

If $a_r(z_0) \neq 0$, analytic solutions (in the neiborhood of z_0) of

$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0$$

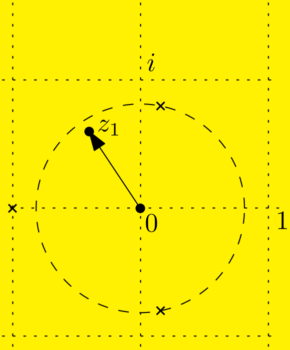
form an r-dimensional vector space.
Moreover, their Taylor series in z_0 converge (at least) in a disk extending to the nearest zero of a_r.

Arbitrary D-finite Functions

A Random Example

$$(z+1)(3z^2-z+2)\,y''' + (5z^3+4z^2+2z+4)y'' \\ +(z+1)(4z^2+z+2)y' + (4z^3+2z^2+5)y = 0$$

$$y(0)=0, y'(0)=i, y''(0)=0$$



$$y(z_1) \simeq -0{,}5688220713892109968232887489539\\
40401816728372266594043883320346219592758\\
12320494797058201136707120728488174753296\\
40179618640233165335353913821228176742066\\
38746845195076195216482627052648481989147\\
-0{,}41951120825888216814674495005568322636\\
04890369475390958159560577151580169021584\\
69436992399704818660023662419290957376458\\
10730416775833847769588392648233263560262\\
18036663454753771692569046113725631\ i$$

$$z_1 = \frac{-2+3i}{5}$$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

$$a_r(z)\,y^{(r)}(z) + \cdots + a_1(z)\,y'(z) + a_0(z)\,y(z) = 0 \qquad a_r(0) \neq 0$$

$$y(z) = \sum_n y_n z^n \qquad S_n(z) = \sum_{k=0}^{n-1} y_n z^n$$

- Recurrence for the Taylor coefficients
  - Indeterminate coefficients:

$$\begin{aligned} y(z) &= \sum_{n=0}^{\infty} y_n z^n \\ \frac{d}{dz}\, y(z) &= \sum_n (n+1) y_{n+1} z^n \\ z \cdot y(z) &= \sum_n y_{n-1} z^n \end{aligned}$$

$$b_s(n)\,y_{n+s} + \cdots + b_0(n)\,y_n = 0$$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

$$\boxed{a_r(z)\,y^{(r)}(z) + \cdots + a_1(z)\,y'(z) + a_0(z)\,y(z) = 0 \qquad a_r(0) \neq 0}$$

$$y(z) = \textstyle\sum_n y_n z^n \qquad S_n(z) = \textstyle\sum_{k=0}^{n-1} y_n z^n$$

- Recurrence for the Taylor coefficients
  $b_s(n)\,y_{n+s} + \cdots + b_0(n)\,y_n = 0$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

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$$y(z) = \sum_n y_n z^n \qquad S_n(z) = \sum_{k=0}^{n-1} y_n z^n$$

- Recurrence for the Taylor coefficients
  $b_s(n)\,y_{n+s} + \cdots + b_0(n)\,y_n = 0$
- Recurrence for the coefficients:
  $b_s(n)y_{n+s} \quad + \quad b_{s-1}(n)y_{n+s-1} \quad + \cdots + \quad b_0(n)y_n \quad = 0$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0 \qquad a_r(0) \neq 0$$

$$y(z) = \textstyle\sum_n y_n z^n \qquad S_n(z) = \sum_{k=0}^{n-1} y_n z^n$$

- Recurrence for the Taylor coefficients
  $b_s(n)\, y_{n+s} + \cdots + b_0(n)\, y_n = 0$ $\quad \times z^{n+s}$
- Recurrence for the terms of the sum:
  $b_s(n) y_{n+s} z^{n+s} + z b_{s-1}(n) y_{n+s-1} z^{n+s-1} + \cdots + z^s b_0(n) y_n z^n = 0$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0 \qquad a_r(0) \neq 0$$

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- Recurrence for the Taylor coefficients
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- Recurrence for the partial sums :
  $S_{n+1}(z) - S_n(z) = y_n\, z^n$

Algorithm

Evaluation of D-finite Functions Inside their Disk of Convergence

$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0 \qquad a_r(0) \neq 0$$

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- Recurrence for the Taylor coefficients
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- Recurrence for the terms of the sum:
  $b_s(n) y_{n+s} z^{n+s} + z b_{s-1}(n) y_{n+s-1} z^{n+s-1} + \cdots + z^s b_0(n) y_n z^n = 0$
- Recurrence for the partial sums :
  $S_{n+1}(z) - S_n(z) = y_n\, z^n$
- Matrix form, binary splitting

Complexity

How Many Terms Do We Need?

- Goal: $\left|y(z) - \sum_{n=0}^{N-1} y_n z^n\right| \leq 10^{-d}$
- If $|y_n| \leq \alpha^n \overbrace{\phi(n)}^{\exp o(n)}$ then $\left|\sum_{n=N}^{\infty} y_n z^n\right| \leq |\alpha z|^N \overbrace{\sum_{n=0}^{\infty} \phi(N+n)|\alpha z|^n}^{\exp o(N)}$
- Convergence radius: $\rho = 1/\limsup_{n\to\infty} |y_n|^{1/n}$
  $\implies$ best possible $\alpha = 1/\rho$
- Conclusion: $N \simeq \dfrac{d}{\log(\rho/|z|)}$

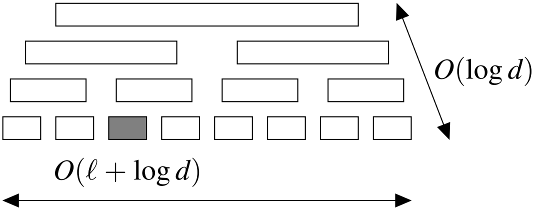
(And we can actually compute such an N.)

Complexity

Binary Splitting

When computing $y(z_1)$, the final recurrence involves z_1

$\ell = \mathrm{size}(z_1)$



$$\sim \frac{d}{\log(\rho/|z|)} \text{ factors}$$

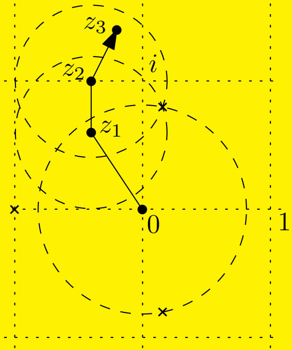
$$\mathrm{M}\left(\frac{d\ (\ell + \log d)}{\log(\rho/|z|)}\right) \log d = \begin{cases} O\big(\mathrm{M}(d \log^2 d)\big) & \text{if } \ell = O(\log d) \\ \Omega(n^2) & \text{if } \ell = d \end{cases}$$

Limitations: $|z_1| < \rho; \quad \ell = O(\log d)$

Numerical Analytic Continuation

$$(z+1)(3z^2-z+2)\,y''' + (5z^3+4z^2+2z+4)y''$$
$$+(z+1)(4z^2+z+2)y' + (4z^3+2z^2+5)y = 0$$

$$y(0)=0,\, y'(0)=i,\, y''(0)=0$$



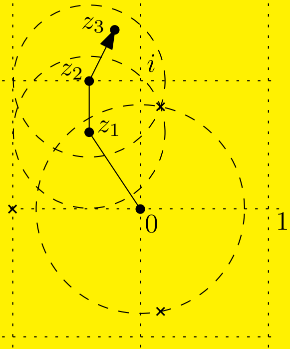
$y(z_3) \simeq -1{,}5598481440603221187326507993405$
$93389341334664487959500453706337545990130$
$23595723610120655516690697098992400952293$
$02516117147544713452845642644966476254288$
$76662237635657163415131886063430803161039$
$-\ 0.71077649435126718436732868786933143977$
$59047479618104045777076954591551406949345$
$14336874295533356649869509377592841606239$
$84373919434109735084282549387411069877437$
$70372320294299156084733705293726504\ i$

$$z_3 = \frac{-1+7i}{5}$$

Transition Matrices (Between Ordinary Points)

$$(z+1)(3z^2 - z + 2)\,y''' + (5z^3 + 4z^2 + 2z + 4)y'' \\ +(z+1)(4z^2 + z + 2)y' + (4z^3 + 2z^2 + 5)y = 0$$

$$\begin{bmatrix} y(z_3) \\ y'(z_3) \\ y''(z_3) \end{bmatrix} = \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{\begin{bmatrix} 1.229919181 + 1.222484838i & -0.710776494 + 1.559848144i & -1.680450593 + 0.8612944465i \\ 2.192415163 - 0.982260350i & 1.428307159 + 1.237636972i & 1.683681888 + 1.443224767i \\ -0.810105380 - 0.813018670i & 0.949416034 - 0.368995278i & -0.309094585 - 0.032241130i \end{bmatrix}} \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix}$$



$$z_3 = \frac{-1+7i}{5}$$

Effective Analytic Continuation

- Solution basis at z_0

$$y_{[z_0,j]}(z) = (z-z_0)^j + \square \cdot (z-z_0)^r + \cdots \qquad j \in [\![0, r-1]\!]$$

- Transition matrix

$$M_{z_0 \to z_1} = \begin{bmatrix} y_{[z_0,0]}(z_1) & \cdots & y_{[z_0,r-1]}(z_1) \\ y'_{[z_0,0]}(z_1) & \cdots & y'_{[z_0,r-1]}(z_1) \\ \vdots & & \vdots \\ \frac{1}{(r-1)!} y^{(r-1)}_{[z_0,0]}(z_1) & \cdots & \frac{1}{(r-1)!} y^{(r-1)}_{[z_0,r-1]}(z_1) \end{bmatrix}$$

- Composition of transition matrices
  = analytic continuation

$$M_{z_0 \to z_1 \to \cdots \to z_m} = M_{z_{m-1} \to z_m} \cdots M_{z_1 \to z_2} \cdot M_{z_0 \to z_1}$$

Points of Large Bit Size

$$\begin{aligned}\mathrm{erf}(\pi) \simeq\ & 0.99999112385363235839473162078120294471238208 15\\ & 128765990475863916467843942619649846027850454178 2613310\\ & 06043264821520306604411963875854074893943387291429 16313\\ & 2555230902334047429212609807578643285046857228864728035\\ & 3074866062036004350772927038034048195719630178507694248\\ & 4951063443190106356178078634699387973616755577593078576\\ & 7867193730580658008654893571733600902958925087790354763\\ & 1634821321290934135517729080384812555377261445353232562\\ & 6651433607961144658060331385205962860463925296434774976\\ & 4667106060908609383010103929356543447438130957966770981\\ & 9560099884058213492947592606412648383713291083934904913\\ & 3976893748259243076371780227275937091363807381587573107\end{aligned}$$

(Bounds not fully implemented yet for this case)

The "Bit Burst" Algorithm

Analytic continuation along

$$
\begin{aligned}
z_0 = 0 &\to z_1 = 0.a_1 \\
&\to z_2 = 0.a_1a_2a_3 \\
&\to z_3 = 0.a_1a_2a_3a_4a_5a_6a_7 \\
&\to z_4 = 0.a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15} \\
&\to \dots \\
&\to z \;\; = 0.a_1a_2 \dots\dots\dots\dots a_n
\end{aligned}
$$

$$|z_{j+1} - z_j| \leq 2^{2^{-j}}$$

Step j $\quad O\left(\mathrm{M}\left(\frac{n\,(\ell + \log n)}{\log(\rho/|\delta z|)} \log n\right)\right) \qquad \begin{cases} \ell = O(2^j) \\ |\delta z| \leq 2^{2^{-j}} \end{cases}$

Total cost $\quad O\left(\sum_{j=0}^{O(\log n)} \mathrm{M}\left(\frac{n\,(2^j + \log n)}{2^j} \log n\right)\right) = O(M(n \log^2 n))$

Some Remarks on Constant Factors

Constant Factors

- At each node of the binary splitting tree, we are multiplying matrices with coefficients in $\mathbb{Z}$ / $\mathbb{Q}$ / $\mathbb{Q}(i)$ / ...

  (or actually elements of any [torsion-free] module-finite $\mathbb{Z}$-algebra)

- In the end the whole computation reduces to additions and multiplications of (huge) integers

- To improve the complexity by a constant factor: do less multiplications
  - "Constant": we regard the order of the recurrence (and thus of the matrices) as fixed
  - $M(n) \gg n \implies$ trade "actual" multiplications for additions / multiplications by constants

  (choose a nice algebra to work in and find an algorithm of low quadratic complexity for this algebra)

A First Example

Spare 20% on Binary Splitting in $\mathbb{Q}(i)$

Karatsuba :

$$(x+iy)(x'+iy') = (u-v) + i(w-u-v)$$

$$\text{where} \quad \begin{cases} u = xx' \\ v = yy' \\ w = (x+y)(x'+y') \end{cases}$$

3 + 1 (denominators) = 4 multiplications instead of 5

(More generally, for $\mathbb{K}$ of characteristic 0, we can multiply elements of $\mathbb{K}[X]/\langle Q \rangle$ using $2 \deg Q - 1$ multiplications in $\mathbb{K}$ [Toom-Cook].)

Matrix Multiplication

- Theory: $O(s^\omega)$, where $\omega < 2.376$ (Coppersmith-Winograd)
- "Practical" for $s > 10^{50}$ or 10^{100}...

- We are interested in fast (less multiplications) algorithms for small sizes
- Usual "bilinear" algorithms work over any ring
- Commutative ring $\implies$ we may also use "quadratic" algorithms

- Classical question
- Already for 3×3 the best bilinear / quadratic algorithms are not known

Multiplication of Small Matrices

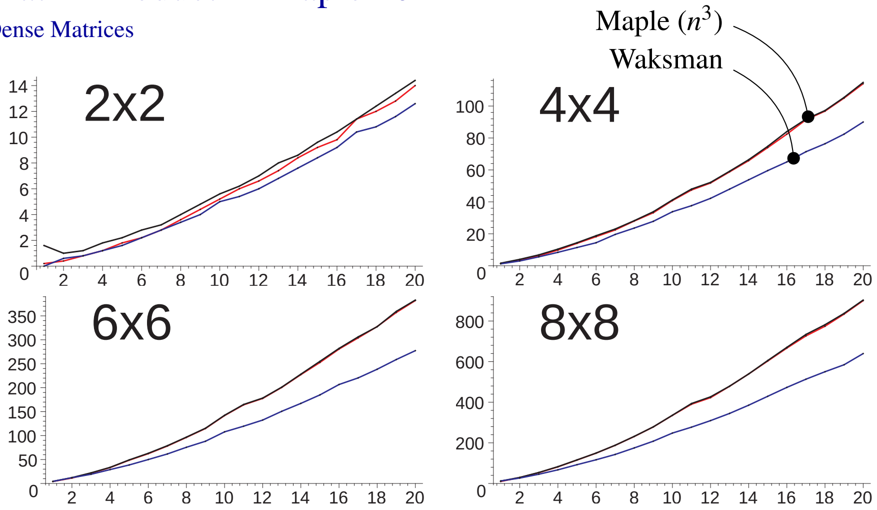
Size	2	3	4	5	6	7	8	9	10
Naive	8	27	64	125	216	343	512	729	1000
NCom	**7**	23	49	100	161	273	343	529	700
Com	**7**	**22**	**46**	**93**	**141**	**235**	**316**	**473**	**595**

Size	11	12	13	14	15	16	17	18	19
Naive	1331	1728	2197	2744	3375	4096	4913	5832	6859
NCom	992	1125	1580	1778	2300	2401	3218	3342	4369
Com	**831**	**987**	**1333**	**1561**	**2003**	**2212**	**2865**	**3231**	**3943**

- Strassen 1977:
  2×2 in 7 (non commutative) mul.
- Waksman 1970:
  $n \times n$ in $n^2 \left\lceil \frac{n}{2} \right\rceil + (2n-1) \left\lfloor \frac{n}{2} \right\rfloor \simeq \frac{n^3}{2} + n^2 - \frac{n}{2}$ commutative mul.

Matrix Product in Maple 10

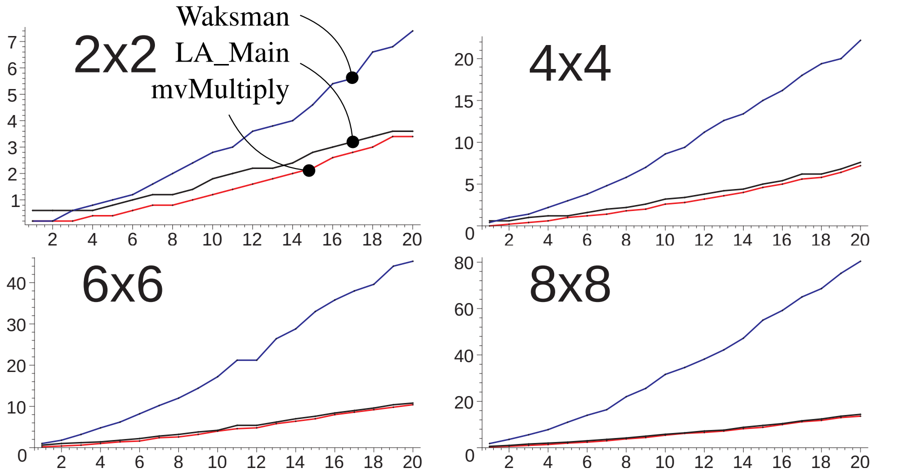
Dense Matrices



Entry size (1,000's of decimal digits) / Time (arbitrary unit)

Matrix Product in Maple 10

Companion Matrices



Entry size (1,000's of decimal digits) / Time (arbitrary unit)

An Alternative Matrix Form for Recurrences

Assume $L = L_k \cdots L_1$ with $L_j = S^{r_j} - c_{r_j-1}^{[j]} S^{r_j-1} - \cdots - c_0^{[j]}$

$$L \cdot u = 0$$

$$u^{[1]} = L_1 \cdot u \qquad u_{n+r_0} = c_0^{[0]} u_n + \cdots + c_{r_0-1}^{[0]} u_{n+r_0-1} + u_n^{[1]}$$

$$u^{[2]} = L_2 \cdot u^{[1]} \qquad u_{n+r_1}^{[1]} = c_0^{[1]} u_n^{[1]} + \cdots + c_{r_1-1}^{[1]} u_{n+r_1-1}^{[1]} + u_n^{[0]}$$

$$\vdots$$

$$u^{[k]} = L_k \cdot u^{[k-1]} \qquad u_{n+r_k}^{[k]} = c_0^{[k]} u_n^{[k]} + \cdots + c_{r_k-1}^{[k]} u_{n+r_k-1}^{[k]} = 0$$

$$= 0$$

Example: for partial sums of P-recursive $L = (S-1)L'$

$$
\begin{bmatrix} u_{n+1}^{[k-1]} \\ \vdots \\ u_{n+r_{k-1}}^{[k-1]} \\ \vdots \\ u_{n+1}^{[1]} \\ \vdots \\ u_{n+r_1}^{[1]} \\ u_{n+1}^{[0]} \\ \vdots \\ u_{n+r_0}^{[0]} \end{bmatrix}
=
\left[\begin{array}{c|c|c|c}
C_{k-1} & & & \\
\hline
 & \ddots & & \\
\hline
 & & C_1 & \\
 & 1 & & \\
\hline
 & & & C_0 \\
 & & 1 &
\end{array}\right]
\begin{bmatrix} u_{n}^{[k-1]} \\ \vdots \\ u_{n+r_{k-1}-1}^{[k-1]} \\ \vdots \\ u_{n}^{[1]} \\ \vdots \\ u_{n+r_1-1}^{[1]} \\ u_{n}^{[0]} \\ \vdots \\ u_{n+r_0-1}^{[0]} \end{bmatrix}
$$

Summary

Fast integer multiplication

+ Two nice algorithmic ideas (binary splitting, bit burst)

+ Bounds

$\longrightarrow$ Fast high-precision analytic continuation

Code available

Some questions

- More efficient unrolling w.r.t. the order of the recurrence?
- $n!$ may be computed in time $O(\mathrm{M}(n \log n))$ [Schönhage].
  Does that generalize to more P-recursive sequences?
- For $s = 2, 3, 4 \ldots$, what is the minimal number of commutative scalar multiplications needed to multiply $s \times s$ matrices?
- Definite integrals of D-finite functions?
- Efficient multipoint evaluation of D-finite functions?
- …

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Thank you!