Evaluation of $A_i(x)$ with Reduced Cancellation

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The Airy Function $\text{Ai}(x)$

\[ \text{Ai}''(x) = x \, \text{Ai}(x) \]

\[ \text{Ai}(0) = \frac{1}{3^{2/3} \Gamma(2/3)} \quad \text{Ai}'(0) = -\frac{1}{3^{1/3} \Gamma(1/3)} \]
Multiple-Precision Evaluation for $x > 0$

**Standard Approach**

**“Small” $x$:**
- Taylor Series at 0
- Catastrophic cancellation for moderately large $x$
- Need $p_{\text{work}} \gg p_{\text{res}}$

**“Large” $x$:**
- Asymptotic Expansion at $\infty$

**This talk**

New evaluation algorithm for “small” $x$ with $p_{\text{work}} \approx p_{\text{res}}$
Complete error analysis
Catastrophic Cancellation

\[ \text{lost digits} \approx \log \left( \max |y_n x^n| \right) - \log |y(x)| \]

\[ \text{Ai}(x) = A - B x + \frac{A}{6} x^3 - \frac{B}{12} x^4 + \frac{A}{180} x^6 - \frac{B}{504} x^7 + \frac{A}{12960} x^9 - \ldots \]
Another Example
The Error Function

\[
erf(x) = \frac{2}{\sqrt{\pi}} \left( x - \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{42} x^7 + \frac{1}{216} x^9 - \ldots \right)
\]

catastrophic cancellation

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Algorithm

1. Compute \[ \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{1 \cdot 3 \ldots (2n+1)} \] positive terms, no cancellation
2. Compute \( \exp(x^2) \)
3. Divide
Where does this formula come from?
The Gawronski-Müller-Reinhard Method

Or: How Complex Analysis “explains” the previous trick

Idea: Find $F$ and $G$ such that

1. $y(x) = \frac{G(x)}{F(x)}$

2. $F$ and $G$ computable with little cancellation


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Asymptotics

\[ \text{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}z^{3/2}\right)}{2\sqrt{\pi}z^{1/4}} \]

as \( z \to \infty \)

in any sector

\( \{z \in \mathbb{C} | -\varphi < \arg z < \varphi\} \)

with \( \varphi > 0 \)
Asymptotics

\[ |\text{Ai}(re^{i\theta})| \approx \exp(h(\theta) r^\rho) \]
for large \( r \)

\[ \text{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3} z^{3/2}\right)}{2 \sqrt{\pi} z^{1/4}} \]

**Order** \( \rho = 3/2 \)

**Indicator** \( h(\theta) = -\frac{2}{3} \cos \frac{3 \theta}{2} \)
Lost in Cancellation

\[
\text{lost digits} \approx \log \left( \max_n |y_n(r e^{i\theta})^n| \right) - \log |y(r e^{i\theta})| \approx r^\rho \left( \max h - h(\theta) \right)
\]
The GMR Method

\[
\begin{align*}
|F(z)| & \approx \exp(h_F(\theta) \cdot r^\rho) \\
|G(z)| & \approx \exp(h_G(\theta) \cdot r^\rho)
\end{align*}
\Rightarrow
\left| \frac{G(z)}{F(z)} \right| \approx \exp \left( \frac{(h_G(\theta) - h_F(\theta)) \cdot r^\rho}{h_{G/F}(\theta)} \right)
\]

Idea (refined): look for
- an auxiliary series \( F \),
- a modified series \( G = y \cdot F \),
both of order \( \rho \),
such that \( h_F \) and \( h_G \) \( \approx \) their max for \( \theta = 0 \)
**Indicators**

\[ F(x) = \text{Ai}(j x) \text{Ai}(j^{-1} x) \]
\[ G(x) = \text{Ai}(x) \text{Ai}(j x) \text{Ai}(j^{-1} x) \]
How do we evaluate the auxiliary series?
A function $y$ is **D-finite** (holonomic) when it satisfies a linear ODE with polynomial coefficients.

**Examples:** $\text{Ai}(x)$, $\exp(x)$, $\text{erf}(x)$…

If $f(x)$, $g(x)$ are D-finite, then:

- $f(x) + g(x)$ and $f(x) \cdot g(x)$ too
  
  $$F(x) = \text{Ai}(jx) \cdot \text{Ai}(j^{-1}x)$$

  $$F'''(x) = 4x F'(x) + 2 F(x)$$

- The **Taylor coefficients** of $f(x)$ obey a linear recurrence relation with polynomial coefficients

  $$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

  $$F_{n+3} = \frac{2(2n+1)(n+2)(n+3)}{(n+1)(n+2)(n+3)} F_n$$
The Auxiliary Series $F(x)$

**D-Finiteness**

\[ F_{n+3} = \frac{2 (2n + 1)}{(n+1)(n+2)(n+3)} F_n \]

\[ F_0 = \frac{1}{3^{4/3} \Gamma\left(\frac{2}{3}\right)^2} \quad F_1 = \frac{1}{2 \sqrt{3} \pi} \quad F_2 = \frac{1}{3^{2/3} \Gamma\left(\frac{1}{3}\right)^2} \]

- Two-term recurrence $\Rightarrow$ Easy to evaluate
- Obviously $F_n > 0$ $\Rightarrow$ Minimal cancellation
The Modified Series $G(x)$

\[ G(x) = \text{Ai}(x) \ F(x) = \sum_{n=0}^{\infty} G_n \ x^{3n} \]

D-Finiteness

\[ G_{n+2} = \frac{10 (n+1)^2 G_{n+1} - G_n}{(n+1)(n+2)(3n+4)(3n+5)} \]

\[ G_0 = \frac{1}{9 \Gamma\left(\frac{2}{3}\right)^3} \quad G_1 = \frac{1}{18 \Gamma\left(\frac{2}{3}\right)^3} - \frac{1}{3 \Gamma\left(\frac{1}{3}\right)^3} \]

\[ G(x) = 0.44749 \cdot 10^{-1} + 0.50371 \cdot 10^{-2} x^3 + 0.14053 \cdot 10^{-3} x^6 \]
\[ \quad + 0.17388 \cdot 10^{-5} x^9 + 0.12091 \cdot 10^{-7} x^{12} + 0.53787 \cdot 10^{-10} x^{15} + \ldots \]

Observe that $G_n > 0$ (proof?)
The recursive computation of $G_n$ is **unstable**

($G_n$ is a minimal solution of the recurrence)

The computation of the sum $\sum_{n=0}^{\infty} G_n x^n$ is stable (no cancellation)
Miller’s \textbf{backward recurrence} method allows one to compute minimal solutions in a numerically stable way

**Final Algorithm**

1. Compute error bounds, choose working precision \hfill (how?)
2. Compute $F(x)$ by direct recurrence
3. Compute $G(x)$ using Miller’s method
4. Divide

Numerically stable in practice \hfill (proof?)
I didn’t actually prove anything
Making the Analysis Rigorous

- Prove that \((G_n)\) is a minimal solution
  \[\Rightarrow\] Miller’s method works
- Prove that \(G_n \geq 0\)
  \[\Rightarrow\] no cancellation

Main issue:
need bounds on \(G_n\)

- Bound the tails of the series \(F\) and \(G\)
- Bound the roundoff errors in \(\sum F_n x^n\)
- Bound the method error of Miller’s algorithm
- Bound additional roundoff errors due to Miller’s method [M&vdS 1976]

R.M.M. Matthiej & A. van der Sluis, Numerische Mathematik, 1976
Controlling $G_n$

Main Technical Lemma

$G_n \sim \gamma_n = \frac{1}{4 \sqrt{3} \pi 9^n n!^2}$ with $\left| \frac{G_n}{\gamma_n} - 1 \right| \leq 2.4 n^{-1/4}$ for all $n \geq 1$

Corollary: $G_n > 0$ (for large $n$, then for all $n$)

Idea of the proof

- $G_n = \frac{1}{2 \pi i} \oint \frac{G(z)}{z^{3n+1}} \, dz$
- saddle-point method
- $\text{Ai}(z) \sim \frac{e^{-2/3 z^{3/2}}}{2 \sqrt{\pi} z^{1/4}} + \text{error bound}$
Conclusion

Summary

- New well-conditioned formula for $A_i(x)$, obtained by an extension of the GMR method
- Rigorous error analysis on this example
- Ready-to-use multiple-precision algorithm for $A_i(x)$

implementation & suppl. material at http://hal.inria.fr/hal-00767085

Next question: How much of this is specific to $A_i(x)$?

- Entire function
- Ability to find auxiliary series
- D-finiteness [constraints on the order of the recurrences?]
- Asymptotic estimate with error bound
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