# Evaluation of Ai(x) with Reduced Cancellation

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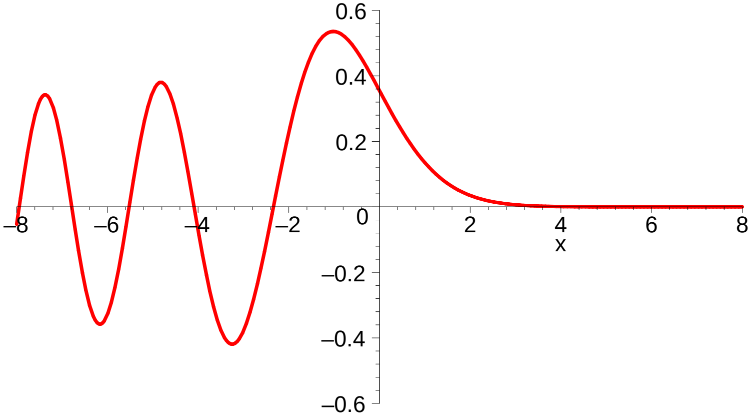
# The Airy Function $\mathrm{Ai}(x)$




$$\mathbf{Ai''(x) = x\,Ai(x)} \qquad \mathrm{Ai}(0) = \frac{1}{3^{2/3}\,\Gamma(2/3)} \qquad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\,\Gamma(1/3)}$$

# Multiple-Precision Evaluation for $x > 0$

## Standard Approach

**"Small" $x$:**

Taylor Series at 0

- catastrophic cancellation for moderately large $x$
- need $p_{\mathrm{work}} \gg p_{\mathrm{res}}$

**"Large" $x$:**

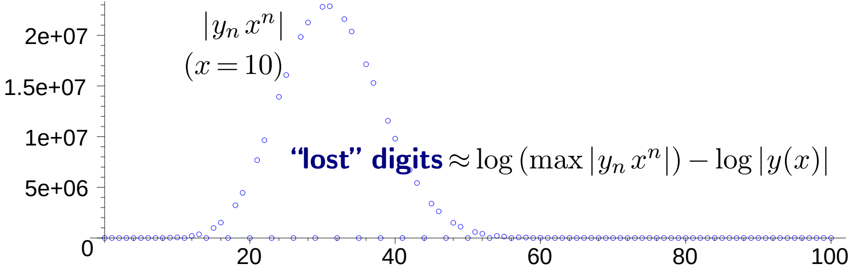
Asymptotic Expansion at $\infty$

## This talk

New evaluation algorithm for "small" $x$ with $p_{\mathrm{work}} \approx p_{\mathrm{res}}$

Complete error analysis

# Catastrophic Cancellation

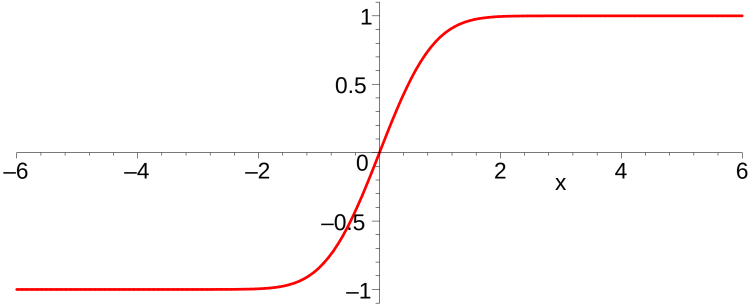



**"lost" digits** $\approx \log\left(\max|y_n\,x^n|\right) - \log|y(x)|$

$$\mathrm{Ai}(x) = A - B\,x + \frac{A}{6}\,x^3 - \frac{B}{12}\,x^4 + \frac{A}{180}\,x^6 - \frac{B}{504}\,x^7 + \frac{A}{12960}x^9 - \cdots$$

# Another Example

## The Error Function




$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left( x - \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{42} x^7 + \frac{1}{216} x^9 - \cdots \right)$$

**catastrophic cancellation**

# But...

**Series Expansions**

$$\text{7.1.5} \quad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

$$\text{7.1.6} \quad = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \ldots (2n+1)} z^{2n+1}$$

[Abramowitz & Stegun 1972, p. 297]

## Algorithm

1. Compute $\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{1 \cdot 3 \cdots (2n+1)}$ positive terms, no cancellation
2. Compute $\exp(x^2)$
3. Divide

# Where

## does this formula

# come from?

# The Gawronski-Müller-Reinhard Method

**Or:** How Complex Analysis "explains" the previous trick

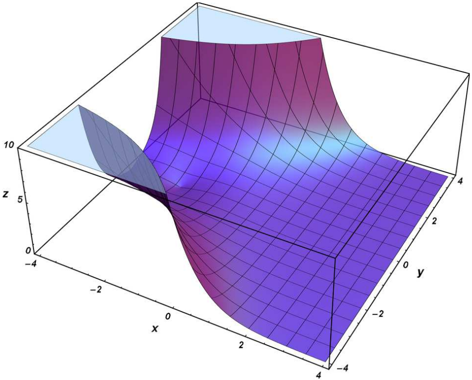
Idea: **Find $F$ and $G$** such that

1. $y(x) = \dfrac{G(x)}{F(x)}$
2. $F$ and $G$ computable with little cancellation

W. Gawronski, J. Müller, M. Reinhard. *SIAM J. Num. An.*, 2007

M. Reinhard. Phd thesis, Universität Trier, 2008

# Asymptotics




$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}\, z^{3/2}\right)}{2\sqrt{\pi}\, z^{1/4}}$$

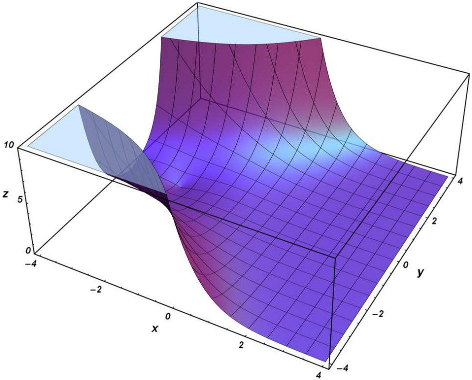
as $z \to \infty$

in any sector

$\{z \in \mathbb{C} | -\varphi < \arg z < \varphi\}$

with $\varphi > 0$

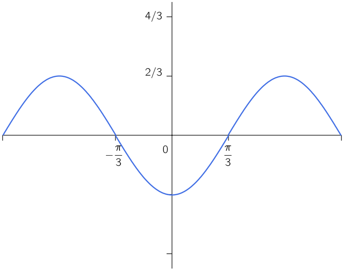
# Asymptotics




$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}\,z^{3/2}\right)}{2\sqrt{\pi}\,z^{1/4}}$$

**Order** $\rho = 3/2$

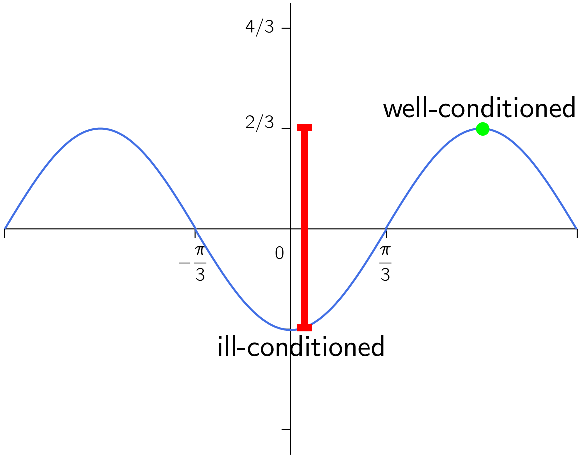
**Indicator** $h(\theta) = -\dfrac{2}{3}\cos\dfrac{3\,\theta}{2}$




$$|\mathrm{Ai}(r\,e^{i\theta})| \approx \exp\left(\boldsymbol{h}(\theta)\,r^{\boldsymbol{\rho}}\right)$$

for large $r$

# Lost in Cancellation




$$\textbf{lost digits} \approx \log\left(\max_n |y_n\,(r\,e^{i\theta})^n|\right) - \log|y(r\,e^{i\theta})| \approx r^\rho\,(\boldsymbol{\max h - h(\theta)})$$

# The GMR Method

$$\begin{cases} |F(z)| \approx \exp\left(\boldsymbol{h_F(\theta)}\, r^\rho\right) \\ |G(z)| \approx \exp\left(\boldsymbol{h_G(\theta)}\, r^\rho\right) \end{cases} \quad \Rightarrow \quad \left|\frac{G(z)}{F(z)}\right| \approx \exp\Big[\underbrace{(\boldsymbol{h_G(\theta)} - \boldsymbol{h_F(\theta)})}_{h_{G/F}(\theta)}\, r^\rho\Big]$$

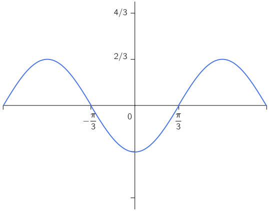
Idea (refined): look for

- an auxiliary series $F$,
- a modified series $G = y\,F$,

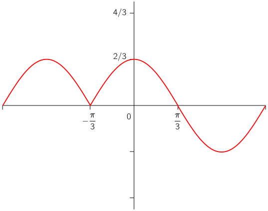
both of order $\rho$,

such that $h_F$ and $h_G \approx$ their max for $\theta = 0$

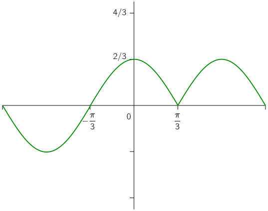
# Indicators



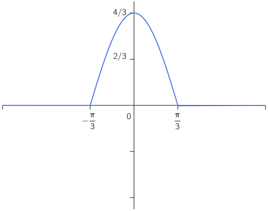

$\mathrm{Ai}(x)$



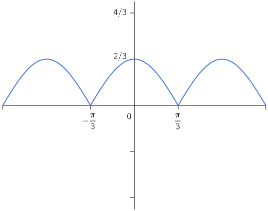

$\mathrm{Ai}(j^{-1}\,x)$




$\mathrm{Ai}(j\,x)$




$\boldsymbol{F(x)} = \mathrm{Ai}(j\,x)\,\mathrm{Ai}(j^{-1}\,x)$




$\boldsymbol{G(x)} = \mathrm{Ai}(x)\,\mathrm{Ai}(j\,x)\,\mathrm{Ai}(j^{-1}\,x)$

**How do we**

# evaluate

**the auxiliary series?**

# Computer Algebra to the Rescue

A function $y$ is **D-finite** (holonomic) when it satisfies a linear ODE with polynomial coefficients.

**Examples:** $\mathbf{Ai(x)}, \exp(x), \operatorname{erf}(x)...$ $\qquad \operatorname{Ai}''(x) = x \operatorname{Ai}(x)$

If $f(x)$, $g(x)$ are D-finite, then:

- $f(x) + g(x)$ and $f(x) \cdot g(x)$ too

  $F(x) = \operatorname{Ai}(j\,x) \cdot \operatorname{Ai}(j^{-1}\,x)$ $\qquad F'''(x) = 4\,x\,F'(x) + 2\,F(x)$

- The Taylor coefficients of $f(x)$ obey a linear recurrence relation with polynomial coefficients

  $F(x) = \sum_{n=0}^{\infty} F_n\,x^n$ $\qquad F_{n+3} = \frac{2\,(2\,n+1)}{(n+1)\,(n+2)\,(n+3)}\,F_n$

# The Auxiliary Series $F(x)$

## D-Finiteness

$$\boldsymbol{F_{n+3}} = \frac{2\,(2\,n+1)}{(n+1)\,(n+2)\,(n+3)}\,\boldsymbol{F_n}$$

$$F_0 = \frac{1}{3^{4/3}\,\Gamma\left(\frac{2}{3}\right)^2} \qquad F_1 = \frac{1}{2\,\sqrt{3}\,\pi} \qquad F_2 = \frac{1}{3^{2/3}\,\Gamma\left(\frac{1}{3}\right)^2}$$

- Two-term recurrence $\Rightarrow$ Easy to evaluate
- Obviously $F_n > 0$ $\Rightarrow$ Minimal cancellation

# The Modified Series $G(x)$

$$G(x) = \mathrm{Ai}(x)\,F(x) = \sum_{n=0}^{\infty} G_n\,\boldsymbol{x^{3n}}$$

## D-Finiteness

$$\boldsymbol{G_{n+2}} = \frac{10\,(n+1)^2\,\boldsymbol{G_{n+1}} - \boldsymbol{G_n}}{(n+1)\,(n+2)\,(3\,n+4)\,(3\,n+5)}$$

$$G_0 = \frac{1}{9\,\Gamma\left(\frac{2}{3}\right)^3} \qquad G_1 = \frac{1}{18\,\Gamma\left(\frac{2}{3}\right)^3} - \frac{1}{3\,\Gamma\left(\frac{1}{3}\right)^3}$$

$$\begin{aligned} G(x) = {} & 0.44749\cdot 10^{-1} + 0.50371\cdot 10^{-2}\,x^3 + .14053\cdot 10^{-3}\,x^6 \\ & + .17388\,10^{-5}\,x^9 + .12091\cdot 10^{-7}\,x^{12} + .53787\cdot 10^{-10}\,x^{15} + \cdots \end{aligned}$$

Observe that $\boldsymbol{G_n > 0}$ (proof?)

# Bad News

The recursive computation of $G_n$ is

**unstable**

($G_n$ is a minimal solution of the recurrence)

The computation of the **sum** $\sum_{n=0}^{\infty} G_n x^n$ is stable (no cancellation)

# All Is Not Lost

Miller's **backward recurrence** method allows one to compute minimal solutions in a numerically stable way

### Final Algorithm

1. Compute error bounds, choose working precision (how?)
2. Compute $F(x)$ by direct recurrence
3. Compute $G(x)$ using Miller's method
4. Divide

Numerically stable in practice (proof?)

I didn't actually

**prove**

anything

# Making the Analysis Rigorous

- Prove that $(G_n)$ is a minimal solution
  [$\Rightarrow$ Miller's method works]
- Prove that $G_n \geqslant 0$
  [$\Rightarrow$ no cancellation]

**Main issue:**

need **bounds on $G_n$**

- Bound the tails of the series $F$ and $G$
- Bound the roundoff errors in $\sum F_n x^n$
- Bound the method error of Miller's algorithm
- Bound additional roundoff errors due to Miller's method [M&vdS 1976]

R.M.M. Matthiej & A. van der Sluis, *Numerische Mathematik*, 1976

# Controlling $G_n$

## Main Technical Lemma

$$G_n \sim \gamma_n = \frac{1}{4\sqrt{3}\,\pi\,9^n\,n!^2} \quad \text{with} \quad \left|\frac{G_n}{\gamma_n} - 1\right| \leqslant 2.4\,n^{-1/4} \quad \text{for all} \quad n \geqslant 1$$

**Corollary:** $G_n > 0$ (for large $n$, then for all $n$)

### Idea of the proof

- $$G_n = \frac{1}{2\pi i}\oint \frac{G(z)}{z^{3n+1}}\,\mathrm{d}z$$
- saddle-point method
- $$\mathrm{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}\,z^{1/4}} + \text{error bound}$$

# Conclusion

## Summary

- New well-conditioned formula for $\mathrm{Ai}(x)$,
  obtained by an extension of the GMR method
- Rigorous error analysis on this example
- Ready-to-use multiple-precision algorithm for $\mathrm{Ai}(x)$
  implementation & suppl. material at `http://hal.inria.fr/hal-00767085`

## Next question: How much of this is specific to $\mathrm{Ai}(x)$?

- Entire function
- Ability to find auxiliary series
- D-finiteness [constraints on the order of the recurrences?]
- Asymptotic estimate with error bound

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