MPRI C-2-22 — Lecture 15

# Solutions of linear differential equations

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# Introduction



In this lecture, $\mathbb{K}$ is an effective field of characteristic zero.

**Problem.** Given a differential equation

$$a_r(x)\,y^{(r)} + \cdots + a_1(x)\,y'(x) + a_0(x)\,y(x) = 0, \qquad a_i \in \mathbb{K}[x],$$

(or a system $Y'(x) = A(x)\,Y(x)$), compute its…

a) formal series solutions $y \in \mathbb{K}[[x]]$,

b) polynomial solutions $y \in \mathbb{K}[x]$,

c) rational solutions $y \in \mathbb{K}(x)$,

d) generalized series solutions,

e) hyperexponential solutions.

Operator notation:

$$a_r(x)\,y^{(r)} + \cdots + a_1(x)\,y'(x) + a_0(x)\,y(x) = 0 \quad \Leftrightarrow \quad (a_r(x)\,D^r + \cdots + a_1(x)\,D + 1)(y) = 0$$

# 1 Differential opearators as skew polynomials

# Differential operators as skew polynomials



Algebraic framework for working with operators $f \mapsto (x \mapsto \sum_i a_i(x)\, f^{(i)}(x))$

**Definition.**

$$\mathbb{K}(x)\langle D\rangle = \left\{ \sum_{i=0}^{r} a_i(x)\, D^i \;\middle|\; \begin{array}{l} r \in \mathbb{N}, \\ a_i \in \mathbb{K}(x) \end{array} \right\}$$

with the usual addition of polynomials,
multiplication defined by $D \cdot x = x \cdot D + 1$ and linearity.

Alt.: $A/(A\,\langle D\,x - x\,D - 1\rangle\,A)$ where $A =$ ring of noncommutative polynomials in $D$ over $\mathbb{K}(x)$.

**Exercise.**

- Compute $D\,(x\,D - 1)$
- Interpret in terms of the solutions of $y' = 0$, $x\,y' = y$, and $y'' = 0$

# Skew Euclidean structure



[Ore 1933, …]

- Euclidean right division:

$$L = Q\,P + R \qquad \text{with } \operatorname{order}(R) < \operatorname{order}(P)$$

- Greatest common right divisor: (↔ common solutions)

$$\begin{cases} L_1 = Q_1\,G \\ L_2 = Q_2\,G \end{cases} \qquad \text{with } G \text{ of max order}$$

- Least common left multiple: (↔ closure by sum)

$$U_1\,L_1 = U_2\,L_2 = M \quad \text{of min order}$$

- Non-commutative Euclidean algorithm
- Annihilating (left) ideal:

$$\begin{aligned} \operatorname{Ann}(f) &= \{L \mid L(f) = 0\} \\ &= \mathbb{K}(x)\langle D\rangle\,G \quad \text{where } G = \text{minimal annihilator of } f \end{aligned}$$

# Recurrence operators as skew polynomials



**Definition.**

$$\mathbb{K}(n)\langle S\rangle = \left\{ \sum_{i=0}^{s} b_i(n)\, S^i \;\middle|\; \begin{array}{l} s \in \mathbb{N}, \\ b_i \in \mathbb{K}(n) \end{array} \right\}$$

with the usual addition of polynomials,
multiplication defined by $S \cdot n = (n+1) \cdot S$ and linearity.

- Also a skew Euclidean ring
- Diff. eq. $\leftrightarrow$ rec. correspondance:

$$\mathbb{K}[x, x^{-1}]\langle D\rangle \quad \cong \quad \mathbb{K}[n]\langle S, S^{-1}\rangle \qquad \text{by} \begin{cases} x \mapsto S^{-1} \\ D \mapsto (n+1)\, S. \end{cases}$$

# 2 Power series solutions

# Singular points



$$L = a_r D^r + \cdots + a_1 D + a_0$$

**Definition.** A point $\xi \in \mathbb{K}$ is called

- an **ordinary point** of $L$ if $a_r(\xi) \neq 0$,
- a **singular point** of $L$ otherwise.

# Formal Laurent series



**Definition.** We denote

$$\mathbb{K}((x)) = \bigcup_{n_0 \in \mathbb{Z}} \left\{ \sum_{n \geqslant n_0} u_n x^n \,\middle|\, u_n \in \mathbb{K} \right\}.$$

The elements of $\mathbb{K}((x))$ are called **formal Laurent series**.

- Warning: $\mathbb{C}((x)) \neq$ Laurent series from analysis (even when convergent).
  In complex analysis, Laurent series are double-sided: $\sum_{n \in \mathbb{Z}} u_n x^n$.
  But formal double-sided series do not form a ring!

- $\mathbb{K}((x))$ is the field of fractions of $\mathbb{K}[[x]]$.

- Rational functions in $\mathbb{K}(x)$ can be expanded in formal Laurent series at any point of $\mathbb{K}$.

# The Euler derivative



**Lemma.** Let $\theta = x\,D$.

Any differential operator

$$L = a_r(x)\,D^r + \cdots + a_1(x)\,D + a_0(x) \in \mathbb{K}[x]\langle D\rangle$$

can be written

$$L = x^{-k}\,[\underbrace{\tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_1(x)\,\theta + \tilde{a}_0(x)}_{\tilde{L}}], \qquad \tilde{a}_i \in \mathbb{K}[x]$$

for some $k \in \mathbb{N}$.

**Proof.** Substitute $x^{-1}\,\theta$ for $D$ and clear denominators.

(Alternatively, perform repeated right Euclidean divisions by $\theta$.) □

**Remark.** $L$ and $\tilde{L}$ have the same solutions.

# Laurent series solutions and recurrences



For $y \in \mathbb{K}((x))$, define $(y_n)_{n\in\mathbb{Z}}$ by $y(x) = \sum_{n\in\mathbb{Z}} y_n x^n$. (Thus $y_n = 0$ for $n \ll 0$.)

**Proposition.** Let $\theta = x\,D$. The series $y \in \mathbb{K}((x))$ is solution to [$\theta$: $y(x) \mapsto x\,y'(x)$]

$$\tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_1(x)\,\theta + \tilde{a}_0(x)$$

if and only if the sequence $(y_n)_{n\in\mathbb{Z}}$ is solution to [$S^{-1}$: $(u_n)_{n\in\mathbb{Z}} \mapsto (u_{n-1})_{n\in\mathbb{Z}}$]

$$R = \tilde{a}_r(S^{-1})\,n^r + \cdots + \tilde{a}_1(S^{-1})\,n + \tilde{a}_0(S^{-1}).$$

**Proof.** Substitute and compare coefficients. □

**Notation.** Given $R$ as above, we write

$$S^{\delta} R = q_0(n) - q_1(n)\,S^{-1} - \cdots - q_s(n)\,S^{-s}$$

with $\delta \in \mathbb{Z}$ chosen so that $q_0 \neq 0$.

# Solutions of singular recurrences



$$\forall n \in \mathbb{Z}, \quad q_0(n)\, y_n - q_1(n)\, y_{n-1} - \cdots - q_s(n)\, y_{n-s} = 0$$

At a **singular index** (= root of $q_0$):

$$\begin{cases} & \vdots & \\ q_0(n-2)\, y_{n-2} & = & q_1(n-2)\, y_{n-3} + \cdots + q_s(n-2)\, y_{n-2-s} \\ q_0(n-1)\, y_{n-1} & = & q_1(n-1)\, y_{n-2} + \cdots + q_s(n-1)\, y_{n-1-s} \\ 0\, y_n & = & q_1(n) \quad y_{n-1} + \cdots + q_s(n) \quad y_{n-s} \\ q_0(n+1)\, y_{n+1} & = & q_1(n+1)\, y_n \quad + \cdots + q_s(n+1)\, y_{n+1-s} \\ q_0(n+2)\, y_{n+2} & = & q_1(n+2)\, y_{n+1} + \cdots + q_s(n+2)\, y_{n+2-s} \\ & \vdots & \end{cases}$$

$\to$ free choice of $y_n$

$\to$ extra linear constraint on $(y_{n-s}, \ldots, y_{n-1})$

**Observations.**

- For a solution $(y_n)_{n\in\mathbb{Z}} = (\ldots, 0, 0, 0, y_N, y_{N+1}, y_{N+2}, \ldots)$ with $y_N \neq 0$ to exist, $N$ must be a root of $q_0$.
- A partial solution $(y_n)_{n\leqslant N}$ with $N \geqslant \max\{\text{roots of } q_0 \text{ in } \mathbb{Z}\}$ extends to a unique solution $(y_n)_{n\in\mathbb{Z}}$.

# Solutions of singular recurrences



**Lemma.** Let $(y_n)_{n\in\mathbb{Z}}$ be a solution to

$$q_0(n)\, y_n = q_1(n)\, y_{n-1} + \cdots + q_s(n)\, y_{n-s}.$$

Assume that there exists an integer $N$ such that $y_n = 0$ for all $n < N$.

Then the largest $N$ with this property is a root of $q_0$.

**Exercise.** Find the dimension of the space of solutions in $\mathbb{Q}^{\mathbb{Z}}$ of

$$(n-1)\,(n-2)\, u_n = u_{n-1} + (n-2)\, u_{n-2}$$

that are ultimately zero as $n \to -\infty$.

# An exercise



**Exercise.** Find the dimension of the space of solutions in $\mathbb{Q}^{\mathbb{Z}}$ of

$$(n-1)(n-2)\,u_n = u_{n-1} + (n-2)\,u_{n-2}$$

that are ultimately zero as $n \to -\infty$.

**Solution.** The first nonzero term of such a solution must be $u_1$ or $u_2$.

Evaluating the equation at $n=2$ yields $u_1 = 0$.

In contrast, starting from any value of $u_2$, one can define a solution with support $\subseteq \{2, 3, \ldots\}$.

The dimension is $1$.

$$\begin{array}{lcccccc}
 & & & \vdots & & & \\
(n=0) & 2\,u_0 & = & u_{-1} & - & 2\,u_{-2} \\
(n=1) & 0 & = & u_0 & - & u_{-1} \\
(n=2) & 0 & = & u_1 & + & 0\,u_0 \\
(n=3) & 2\,u_3 & = & u_2 & + & u_1 \\
 & & & \vdots & & &
\end{array}$$

**Remark.** This is less than

- the order of the recurrence,
- the number of integer roots of $q_0$

**Remark.** The dimension could also be larger than the order: consider

$$n\,(n-1)\,u_n = (n-1)\,u_{n-1}.$$

# Back to differential equations



$$L = \tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_0(x) \quad \to \quad \begin{aligned} R &= \tilde{a}_r(S^{-1})\,n^r + \cdots + \tilde{a}_0(S^{-1}) \\ &= S^{-\delta}\left(q_0(n) - \cdots - q_s(n)\,S^{-s}\right) \end{aligned}$$

**Definition.** The polynomial $q_0$ is called the **indicial polynomial** of $L$ at $0$.
The polynomial obtained in the same way after $x \leftarrow \xi + x$ is called the indicial polynomial at $\xi$.

**Proposition.** The valuation of any solution $y \in \mathbb{K}((x))$ of $L(y) = 0$ is a root of the indicial polynomial at $0$.

**Corollary.** The space of solutions of $L$ in $\mathbb{K}((x))$ has dimension $\leqslant r$.

**Proof.**

- We can echelonize a basis so that its elements have distinct valuations.
- $\deg_n(S\,n) = \deg_n((n+1)\,S)$ so $\deg_n(R) = r$, and in particular $\deg q_0 \leqslant r$. □

**Remark.** $\deg q_0$ can be $< r$, and the dimension can be < #integer roots of $q_0$

# Computing a basis of formal Laurent series solutions



$$L = \tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_0(x) \qquad \rightarrow \qquad R = \tilde{a}_r(S^{-1})\,n^r + \cdots + \tilde{a}_0(S^{-1})$$

$$(\deg \tilde{a}_i < d) \qquad\qquad = S^{-\delta}\big(q_0(n) - \cdots - q_s(n)\,S^{-s}\big)$$

**Idea.** Let $\lambda, \mu$ be the smallest/largest root of $q_0$ in $\mathbb{Z}$.

- Make an ansatz $y(x) = y_\lambda x^\lambda + \cdots + y_\mu x^\mu + O(x^{\mu+1})$, plug into the equation:

$$L(y)(x) = \underbrace{[\ldots]}\,x^\lambda + \underbrace{[\ldots]}\,x^{\lambda+1} + \underbrace{[\ldots]}_{\text{linear expressions in } y_\lambda, \ldots, y_\mu}\,x^{\mu+d-1} + \underbrace{O(x^{\mu+d})}_{\substack{\text{partial solutions are guaranteed}\\ \text{to extend so as to make this zero}}}$$

- Solve the resulting linear system

For solutions in $\mathbb{K}[[x]]$: same but restrict $\lambda, \mu$ to $\mathbb{N}$.

**Limitation.** The dimension $\mu - \lambda + 1$ can be exponential in the bit size of the input.
For example the solutions of $x^2 y''(x) = 999999\,x\,y'(x)$ are spanned by $1$ and $x^{10^6}$.

# Computing formal series solutions by recurrence



**Algorithm (sketch).** *Input*: $L$, $N$ *Output*: a basis of $\mathrm{Sol}(L, \mathbb{K}[[x]])$ to precision $N$

1. Convert $L$ to a recurrence. Let $q_0$ be the indicial polynomial.
2. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ be the roots of $q_0$ in $\mathbb{N}$. $m \leqslant$ *diff. eq. order*
3. For $n = \lambda_1, \lambda_1 + 1, \ldots, \max(N, \lambda_m)$: *note the* $\max$
   a. If $n = \lambda_k$ for some $k$:
      i. Set $u_n$ to a new indeterminate $C_k$.
      ii. Evaluate the recurrence at $n$, record the resulting relation on previous $C_k$.
   b. Otherwise compute $u_n$ using the recurrence.
4. Solve the linear system on $C_1, \ldots, C_m$ consisting of the collected relations.

Even better: use fast algorithms (baby steps-giant steps, binary splitting)
to "jump" from one singular index to the next.

Remark: the $C_k$ that remain free after solving play the role of generalized initial values

# The case of ordinary points



**Remark.** If $0$ is an ordinary point, just like in the analytic case

- the space of power series solutions has dimension exactly $r$, and
- there exists a basis of solutions with valuations $0, 1, \ldots, r-1$.

**Proof sketch.** One can check that the rec. obtained by $L(x, D) \rightsquigarrow \tilde{L}(x, \theta) \rightsquigarrow R(n, S^{-1})$ has the form

$$\begin{aligned} a_r(0)\; n\,(n-1)\cdots(n-r+1)\; u_n \;=\; & \quad [\mathrm{poly}(n)]\; (n-1)\cdots(n-r+1)\; u_{n-1} \\ & + [\mathrm{poly}(n)]\; (n-2)\cdots(n-r+1)\; u_{n-2} \\ & + \cdots \\ & + [\mathrm{poly}(n)]\; (n-r+1)\; u_{n-r+1} \\ & + \cdots \end{aligned}$$

- Since $a_r(0) \neq 0$, the indicial polynomial is $a_r(0)\; n\,(n-1)\cdots(n-r+1)$.
  This shows that the only possible valuations are $0, \ldots, r-1$.
- All partial solutions $(y_0, \ldots, y_k)$ for $k \leqslant r-1$ extend thanks to the shape of the rhs. $\square$

# A paradox?



**Remark.**

$L$ nonsingular of order $r$ and deg $d$ $\quad\leftrightarrow\quad$ $R$ of order $s \leqslant r + d$

usually $s \neq r$

$$V = \mathrm{Sol}_{\mathbb{K}[[x]]}(L) \quad\leftrightarrow\quad W = \{(y_n) \in \mathbb{K}^{\mathbb{Z}} \text{ with } y_n = 0 \text{ for } n < 0\}$$

$\dim V = r$ $\qquad$ $\dim W$ unrelated to $s$

(but related to $r$ via the indicial polynomial)

# 3 Polynomial and rational solutions

# Introduction



$$a_r\, y^{(r)} + \cdots + a_1(x)\, y' + a_0\, y = 0$$

$$\deg(a_i) < d$$

Suppose we had a bound $B$ on the degrees of polynomial solutions.

Make an ansatz: $y(x) = c_0 + c_1\, x + \cdots + c_{D-1}\, x^{B-1}$,

plug into the equation:

$$a_r(x)\, y^{(r)} + \cdots + a_0(x)\, y(x) = \underbrace{[\ldots]} + \underbrace{[\ldots]}\, x + \underbrace{[\ldots]}\, x^{d+B-2}$$

linear expressions in $c_0, \ldots, c_{N-1}$

$\longrightarrow$ $B + d - 1$ linear equations in $B$ unknowns.

**Questions.**

- Compute the degree bound
- Do better than $B^\theta$

# Finite-support solutions of recurrences



A polynomial solution is just a power series solution that terminates
a solution with finite support of the associated recurrence.

**Lemma.** Consider the recurrence

$$\forall n \in \mathbb{Z}, \quad b_s(n)\, y_{n+s} + \cdots + b_1(n)\, y_{n+1} + b_0(n)\, y_n = 0.$$

For a solution

$$(y_n)_{n\in\mathbb{Z}} = (\ldots, y_{N-2}, y_{N-1}, y_N, 0, 0, 0, \ldots) \qquad \text{with } y_N \neq 0$$

to exist, $N$ must be a root of $b_0$.

**Remark.** If now $\begin{aligned} R &= S^{-\delta}\,(q_0(n) - q_1(n)\,S^{-1} - \cdots) && \text{with } q_0 \not\equiv 0 \\ &= S^{\gamma}\,(b_0(n) + b_1(n)\,S + \cdots) && \text{with } b_0 \not\equiv 0, \end{aligned}$

for a solution

$$(y_n)_{n\in\mathbb{Z}} = (\ldots 0, 0, 0, y_\ell, y_{\ell+1} \ldots, y_{h-1}, y_h, 0, 0, 0, \ldots) \quad \text{with} \quad y_\ell, y_h \neq 0$$

to exist, one must have $q_0(\ell) = b_0(h) = 0$.

# Degree bounds



$$L = \tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_0(x) \quad \to \quad \begin{aligned} R &= \tilde{a}_r(S^{-1})\,n^r + \cdots + \tilde{a}_0(S^{-1}) \\ &= S^\gamma\left(b_0(n) + \cdots - b_s(n)\,S^s\right) \\ &\quad \gamma \text{ such that } b_0 \neq 0 \end{aligned}$$

**Definition.** The polynomial $b_0$ is called the **indicial polynomial at infinity** of $L$.

One can check that it is the indicial polynomial at $0$ of the equation obtained by $x \leftarrow x^{-1}$.

**Proposition.** For any solution $y \in \mathbb{K}[x]$ of $L(y) = 0$, the degree of $y$ is a root of $b_0$.

**Remark.** Polynomial solutions can be large! Last week, we found a small diff. eq. annihilating the dense polynomial $(1+x)^{2N}(1+x+x^2)^N$.

# From power series solutions to polynomial solutions



**Lemma.** Let $L = \tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_0(x)$ with $0$ an ordinary point. Let $d = \max_i \deg \tilde{a}_i$.

Let $N \geqslant \max\{r - d - 1,$ all integer roots of the indicial polynomial at $\infty$ of $L\}$.

Then the solutions of $L$ in $\mathbb{K}[x]$ are exactly its solutions $\sum_{n=0}^{\infty} y_n x^n \in \mathbb{K}[[x]]$ such that

$$y_{N+1} = \cdots = y_{N+d} = 0.$$

**Proof.**

$\subseteq$. Any polynomial solution is also a power series solution.

By the degree bound, it has $y_{N+1} = \cdots = y_{N+d} = 0$.

$\supseteq$. The recurrence associated to $L$ has order $< d$.

Since $0$ is an ordinary point, its only possible singular indices are $0, 1, \ldots, r-1$.

So $y_{N+1} = \cdots = y_{N+d} = 0$ implies $y_n = 0$ for all $n > N$. □

**Exercise.** What happens without the assumption that $0$ is an ordinary point?

# Polynomial solutions of differential equations



**Algorithm.** *Input*: $L = \tilde{a}_r(x)\,\theta^r + \cdots + \tilde{a}_0(x) \in \mathbb{K}[x]\langle\theta\rangle$ *Output*: a basis of $\mathrm{Sol}(L, \mathbb{K}[x])$

1. Shift $x$ to reduce to the case where $0$ is an ordinary point.
2. Let $b_0 =$ indicial polynomial at $\infty$ of $L$.
3. If $b_0$ has no root in $\mathbb{N}$, return $\varnothing$. *cf. Lecture 11 — constrains* $\mathbb{K}$

   Otherwise let $d = \max_i \deg \tilde{a}_i$ and $N = \max\{r-d-1, \text{roots of } b_0 \text{ in } \mathbb{N}\}$.
4. Compute a basis $y_1, \dots, y_r$ of sol. in $\mathbb{K}[[x]]$ truncated to order $N+d+1$.
5. Solve

$$\begin{pmatrix} c_1 & \cdots & c_r \end{pmatrix} \begin{pmatrix} y^{[1]}_{N+1} & \cdots & y^{[1]}_{N+d} \\ \vdots & & \vdots \\ y^{[r]}_{N+1} & \cdots & y^{[r]}_{N+d} \end{pmatrix} = 0.$$

6. Return $\{\sum_i c_i y^{[i]} \mid (c_1, \dots, c_r) \in \text{a basis of solutions of this system}\}$.

Cost: $O(r\,d\,N + \mathrm{poly}(r, d))$ ops

# Polynomial solutions: Remarks



- **Alternative method** avoiding the shift:

  Adapt the algorithm for series solutions at singular points

- **Quick existence check** for polynomial solutions when $\mathbb{K} = \mathbb{Q}$:

  Compute the $y^{[j]}_{N+1+i} \bmod p$ for some prime $p$
  using the baby steps-giant steps method from last week.

- More generally, one can compute the **dimension, degrees and selected terms** of a basis of polynomial solutions without computing the solutions in expanded form.

  (Baby steps-giant steps and/or binary splitting).

# Rational solutions of differential equations



$$a_r y^{(r)} + \cdots + a_1(x)\, y' + a_0\, y = 0 \qquad \text{(DEq)}$$

Rational solutions reduce to polynomial solutions given a multiple of the denominator.

**Observation 1.** Any pole $\xi \in \bar{\mathbb{K}}$ of $y$ must be a singular point.

**Observation 2.** If $y$ has a pole of multiplicity $m$ at $\xi \in \bar{\mathbb{K}}$, its series expansion provides a solution in $\bar{\mathbb{K}}((x-\xi))$ of valuation $m$.

**Proposition.** For all $\zeta \in \bar{\mathbb{K}}$ such that $a_r(\zeta) = 0$, let $m_\zeta$ be the smallest root in $\mathbb{Z}_{<0}$ of the indicial polynomial of (DEq) at $\zeta$ (if any, and $m_\zeta = 0$ otherwise).

Then the denominator of any rational solution of (DEq) is divisible by $Q = \prod_\zeta (x-\zeta)^{m_\zeta}$.

Better variant: attach indicial polynomials to *factors* of $a_r$ instead of roots to avoid working over $\bar{\mathbb{K}}$.

**Algorithm.** Compute $Q$ as above, change $y$ to $\frac{w}{Q}$, basis of solutions $w \in \mathbb{K}[x]$.

# An exercise



**Exercise.** Consider the differential equation

$$(x-1)\,y''(x) + (-x+3)\,y'(x) - y(x) = 0. \tag{E}$$

1. Let $y(x) = \sum_{n=-N}^{\infty} y_n\,(x-1)^n$ be a solution of (E). Set $y_n = 0$ for $n < -N$.
   Show that the sequence $(y_n)_{n\in\mathbb{Z}}$ satisfies

$$\forall n \in \mathbb{Z}, \quad (n+1)\,(n+2)\,y_{n+1} = (n+1)\,y_n.$$

2. Find all rational solutions of (E).

# 4 Differential systems

# From differential systems to scalar equations



**Proposition.** Let $Y$ be a solution of the system

$$Y'(x) = A(x)\,Y(x), \qquad A \in \mathbb{K}(x)^{r\times r}.$$

For any vector $K \in \mathbb{K}(x)^r$, the function $K(x)\,Y(x)$ satisfies a scalar differential equation of order $r \leqslant r$ with coefficients in $\mathbb{K}(x)$.

**Corollary.** The entries of $Y$ are D-finite.

**Proof.**

$$\left.\begin{array}{r} K \\ \\ (K' + K\,A) \\ (\square_1' + \square_1\,A) \\ \\ (\square_{r-1}' + \square_{r-1}\,A) \end{array}\right\} \begin{array}{l} r+1 \text{ vectors} \\ \text{in dimension } r \end{array} \quad \Rightarrow \quad a_0\,K\,Y + \cdots + a_r\,(K\,Y)^{(r)} = 0$$

$\square$

# Solutions of differential systems



Assume that there was **no relation** between $K, K' + K\,A, \ldots$ before the $(r+1)$th element of the sequence.

After solving the resulting scalar equation $L(w) = 0$:

$$\underbrace{\begin{pmatrix} — & K & — \\ — & K' + K\,A & — \\ — & \square' + \square\,A & — \\ & \vdots & \\ — & \square' + \square\,A & — \end{pmatrix}}_{\in \mathbb{K}(x)^{r\times r},\ \text{invertible}} \begin{pmatrix} \\ \\ Y \\ \\ \\ \end{pmatrix} = \underbrace{\begin{pmatrix} K\cdot Y \\ (K\cdot Y)' \\ (K\cdot Y)'' \\ \vdots \\ (K\cdot Y)^{(r-1)} \end{pmatrix}}_{\text{known}}$$

For any rational solution of $Y' = A\,Y$, the function $w = K\,Y$ is a rational solution of $L$, so we can compute the rational solutions of $Y' = A\,Y$ from those of $L$.

# The cyclic vector lemma



**Theorem.** There exists a vector $K \in \mathbb{K}(x)^r$ such that the vectors [Cope 1936]

$$K, \quad \nabla K = K' + K\,A, \quad \ldots, \quad \nabla^{r-1} K$$

are linearly independent over $\mathbb{K}(x)$.

**Proof.** For all $i$, the vector $\nabla^i K$ is of the form

$$\nabla^i K = K^{(i)} + K^{(i-1)} Q_{i,i-1} + \cdots + K\,Q_{i,0} \qquad \text{where} \qquad Q_{i,j} = \operatorname{poly}(A, A', \ldots) \in \mathbb{K}(x)^{r\times r}.$$

Let $x_0 \in \mathbb{K} \setminus \{\text{poles of } A\}$. Define $v_0, \ldots, v_{r-1} \in \mathbb{K}^r$ by

$$v_i = \boldsymbol{e}_i - v_{i-1} Q_{i,i-1}(x_0) + \cdots + v_0\, Q_{i,0}(x_0), \qquad (\boldsymbol{e}_i) = \text{canonical basis of } \mathbb{K}(x)^r.$$

Finally choose $K \in \mathbb{K}[x]$ such that $K(x_0) = v_0, \ldots, K^{(r-1)}(x_0) = v_{r-1}$.

Then $\nabla^i K(x_0) = \boldsymbol{e}_i$ for $0 \leqslant i < r$. In particular the $\nabla^i K$ are linearly independent. □

**Remark.** The proof gives an algorithm to compute a suitable $K$.

# 5 Generalized series solutions

# Introduction



$$L = a_r(x)\,D^r + \cdots + a_1(x)\,D + a_0(x)$$

$$q_0 = \text{indicial polynomial at } 0$$

We have seen that, when $0$ is a singular point:

- $\deg q_0$ can be $< r$,
- $\dim \ker_{\mathbb{K}[[x]]} L$ can be $< \#\{\text{integer roots of } q_0\}$

**Question.** Define and compute a "full" basis of "series" solutions at a singular point.

# Non-integer exponents



When $q_0(\lambda) = 0$ for some $\lambda \notin \mathbb{Z}$, look for solutions $x^\lambda f(x)$ with $f(x) \in \mathbb{K}[[x]]$.

**Examples.**

- $L = 2(x-1)\,x\,D + x + 1 \quad \longrightarrow \quad q_0(\lambda) = 2\lambda - 1$

  solution: $y(x) = x^{1/2}(1 + x + x^2 + \cdots)$
- $L = \theta^2 - 2 = x^2 D^2 + x\,D - 1 \quad \longrightarrow \quad q_0(\lambda) = \lambda^2 - 2$ $\qquad \mathbb{K} = \bar{\mathbb{Q}}$

  solutions: $x^{\pm\sqrt{2}}$
- $L = x^2 D^2 + x\,D - 1 + x \quad \longrightarrow \quad q_0(\lambda) = \lambda^2 - 2$ $\qquad \mathbb{K} = \bar{\mathbb{Q}}$

  solutions: $x^{\pm\sqrt{2}}\left(1 + \frac{1}{7}(1 \mp \sqrt{2})\,x + \frac{1}{28}(5 \mp 3\sqrt{2})\,x^2 + \cdots\right)$

**Algorithm.** Same as for Laurent series.

**Remark.** Here $x^\lambda$ with $\lambda \in \mathbb{K}$ denotes *some algebraic object* satisfying the “usual relations”

E.g., start with $\mathbb{K}((x))[e_\lambda]_{\lambda \in \mathbb{K}}$, quotient by the relations $e_0 = 1$, $e_{\lambda+1} = x\,e_\lambda$, and $e_{\lambda+\mu} = e_\lambda\,e_\mu$, and set $e'_\lambda = \lambda\,e_{\lambda-1}$ to obtain a differential ring containing $\mathbb{K}((x))$.

# Multiple indicial roots



[Frobenius 1873, …]

To recover $\deg q_0$ linearly independent solutions,
consider solutions $x^\lambda \big(f_0(x) + f_1(x)\log x + \cdots + f_{r-1}(x)\log(x)^{r-1}\big)$.

Again, $\log(x)$ is just a notation for an element of a differential extension with $\log'(x) = 1/x$.

**Examples.**

- $L = \theta^2 = x^2 D^2 + x D \quad \longrightarrow \quad q_0(\lambda) = \lambda^2$

  solutions spanned by $1$ and $\log(x)$

- $L = x D^2 - D + 1 \quad \longrightarrow \quad q_0(\lambda) = \lambda(\lambda - 2)$ rec: $n(n-2)\, y_n = y_{n-1}$

  solutions: $x^2 - \frac{1}{3}x^3 + \cdots,$

  $(1 + x - \frac{2}{9}x^3 + \cdots) + (-\frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots)\log(x)$

**Algorithm.** Similar as before with structured systems of recurrences.

When hitting a singular index, insert a new $\log(x)^k$ to gain a degree of freedom.

# Multiple indicial roots



$$L = x\,D^2 - D + 1$$

$$\begin{aligned}
y(x) &= \sum u_n\,x^n + \sum v_n\,x^n \log(x)\\
y'(x) &= \sum n\,u_n\,x^{n-1} + \sum n\,v_n\,x^{n-1}\log(x) + \sum v_n\,x^{n-1}\\
&= \sum (n\,u_n + v_n)\,x^{n-1} + \sum n\,v_n\,x^{n-1}\log(x)\\
y''(x) &= \sum \underbrace{[(n-1)\,(n\,u_n + v_n) + n\,v_n]}_{(n^2-n)u_n+(2n-1)v_n}\,x^{n-2} + \sum (n-1)\,n\,v_n\,x^{n-2}\log(x)
\end{aligned}$$

$$Ly(x) = \sum \underbrace{[n\,(n-2)\,u_n + u_{n-1} + 2\,n\,v_n]}_{\mathrm{REC}_0}\,x^{n-1} + \sum \underbrace{[n\,(n-2)\,v_n + v_{n-1}]}_{\mathrm{REC}_1}\,x^{n-1}\log(x)$$

$$\begin{array}{cccccc}
v_n & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \cdots\\
u_n & 1 & 1 & 0 & -\frac{2}{9} & \cdots\\
n & 0 & 1 & 2 & 3 & \cdots
\end{array}$$

# Regular singular points



**Definition.** A singular point where the indicial polynomial has degree $r$ is called a **regular singular point**.

**Proposition.** Let $L \in \mathbb{K}(x)\langle D \rangle$ be an operator of order $r$ with a regular singular point at $0$. Then $L$ admits $r$ linearly independent formal solutions of the form

$$x^{\lambda}\left(f_0(x) + f_1(x)\log x + \cdots + f_K(x)\log(x)^K\right), \qquad f_k \in \mathbb{K}(\lambda)[[x]],$$

with $K < r$ and $\lambda$ among the roots of the indicial polynomial.

**Proposition.** When $\mathbb{K} = \mathbb{C}$, the $f_k$ are convergent power series.

I.e., one obtains a basis of solutions analytic in $\{|z| < \rho\} \setminus (-\rho, 0]$ for some $\rho > 0$.

# Irregular singular points



When $\deg q_0 < r$, several new phenomena.

**Theorem.** Assume that $\mathbb{K}$ is algebraically closed. [Fabry 1885, …]

Any linear differential equation of order $r$ with coefficients in $\mathbb{K}((x))$ admits $r$ linearly independent formal solutions of the form

$$\exp\left(\gamma_\ell x^{-\ell/p} + \cdots + \gamma_1 x^{-1/p}\right) x^\lambda \left(f_0(x^{1/p}) + \cdots + f_{r-1}(x^{1/p}) \log(x)^{r-1}\right)$$

where $p \in \mathbb{N}$, $\gamma_i, \lambda \in \mathbb{K}$, and $f_j \in \mathbb{K}[[x]]$.

The series $f_k$ are typically divergent.

When $\mathbb{K} = \mathbb{C}$, these expansions can still be interpreted as asymptotic expansions of analytic solutions.

# Newton polygons

**Goal.** Find the leading term inside the exponential. $L = \sum_{i,j} a_{i,j} x^j D^i$

Suppose $\quad y(x) = e^{\gamma x^{-\sigma} + \cdots} \cdot x^{\lambda} \cdot (1 + \square x^{\tau})$ $\quad \sigma > 0$

Then
$$
\begin{aligned}
y'(x) &= -\gamma\,\sigma\,x^{-\sigma-1}\,e^{\gamma x^{-\sigma}+\cdots} \cdot x^{\lambda} \cdot (1+\cdots) \\
&\quad + e^{\gamma x^{-\sigma}+\cdots} \cdot \lambda\,x^{\lambda-1} \cdot (1+\cdots) \\
&\quad + e^{\gamma x^{-\sigma}+\cdots} \cdot x^{\lambda} \cdot \left(\square\,x^{\tau-1} + \cdots\right) \\
&= -\gamma\,\sigma\,e^{\gamma x^{-\sigma}+\cdots}\,x^{\lambda-\sigma-1}\,(1+\cdots)
\end{aligned}
$$

$$
x^j D^i \cdot y(x) = (-\gamma\,\sigma)^i\,x^{j-i(\sigma-1)} \cdot e^{\gamma x^{-1/p}+\cdots}\,x^{\lambda}\,(1+\cdots)
$$

Fix $\sigma$. To have $L \cdot y = 0$, the leading $x^{j-i(\sigma-1)}$ (smallest exponent) must be reached **at lest twice** when considering all $a_{i,j}\,x^j\,D^i \cdot y(x)$:

$$
j_1 - i_1\,(\sigma-1) = j_2 - i_2\,(\sigma-1) \qquad \Rightarrow \qquad \sigma - 1 = \frac{j_2 - j_1}{i_2 - i_1}
$$

Additionally the corresponding $a_{i,j}\,(-\gamma\,\sigma)$ must sum to zero (**characteristic equation**).

# Computing all generalized series solutions



$$y(x) = \exp\left( \gamma_\ell x^{-\ell/p} + \cdots + \gamma_1 x^{-1/p} \right) x^\lambda \left( f_0(x^{1/p}) + \cdots + f_{r-1}(x^{1/p}) \log(x)^{r-1} \right)$$

To compute a basis of generalized series solutions:

- Compute solutions with no exponential factor as in the regular case
- Find candidates for $-\ell/p$ using the Newton polygon
- Find candidates for $\gamma_\ell$ using the characteristic equation
- For each candidate, write $y(x) = e^{-\gamma_\ell x^{-\ell/p}} \tilde{y}(x^{1/p})$ and recurse

# 6 Bonus: Hyperexponential solutions, First-order factors

# Hyperexponential functions



**Definition.** A "function" $y(x)$ is called **hyperexponential** over $\mathbb{K}$ when $\dfrac{y'(x)}{y(x)} \in \mathbb{K}(x)$.

Here "function" = analytic function over $\mathbb{C}$, or more generally element of a differential extension of $\mathbb{K}(x)$.

Examples: $\dfrac{x^3+2}{x^2-1}$, $\sqrt{1+x}$, $e^x \dfrac{\sqrt{1+x}}{x^2+1}$

Closed form:

$$\begin{aligned} y(x) &= \exp \int \left[ \frac{\beta_{1,1}}{x-\xi_1} + \frac{\beta_{1,2}}{(x-\xi_1)^2} + \cdots + \frac{\beta_{2,1}}{x-\xi_2} + \cdots \right] \\ &= e^{\mathrm{rat}(x)} \, (x-\xi_1)^{\beta_{1,1}} \, (x-\xi_2)^{\beta_{2,1}} \ldots \end{aligned}$$

**Goal.** Given $L$, compute all hyperexponential solutions.

Note that the set of hyperexponential solutions is not a vector space!

**Proposition.** A function $y$ is a hyperexponential solution of a differential operator $L$ iff $L$ is right-divisible by $D - y'/y$.

# Exponential parts



Consider an hyperexponential function

$$y(x) = e^{\frac{p_1(x)}{(x-\xi_1)^{m_1}} + \frac{p_2(x)}{(x-\xi_2)^{m_2}} + \cdots} (x-\xi_1)^{\beta_{1,1}} (x-\xi_2)^{\beta_{2,1}} \cdots.$$

At $\xi_i$:

$$y(\xi_i + z) = e^{\gamma_{m_i} z^{-m_i} + \cdots + \gamma_1 z^{-1}} z^{\beta_{i,1}} (\square + \square z + \cdots). \qquad (*)$$

If $L \cdot y = 0$, this expansion must be among the generalized series solutions of $L$ at $\xi_i$.

**Definition.** We call:

- **local exponential parts** of $L$ at $\xi$ the factors $\exp\left(\gamma_\ell z^{-\ell/p} + \cdots + \gamma_1 z^{-1/p}\right) z^\lambda$ appearing in generalized series solutions in $z = x - \xi$, considered up integer powers of $z$;
- **local exponential part** of $y$ at $\xi$ the corresponding factor in $(*)$;
- **global exponential part** of $y$ its equivalence class for the relation

$$y_1 \sim y_2 \Leftrightarrow \frac{y_1}{y_2} \in \mathbb{K}(x).$$

# The classical algorithm



[Fabry 1885]

Idea:
- the collection of local exponential parts of $y$ at every $\xi$ determines its global exponential part;
- for $y$ to be a solution of $L$, the local exponential parts of $y$ at every $\xi$ must be among those of $L$.

**Algorithm.** *Input*: $L \in \mathbb{K}(x)\langle D \rangle$ *Output*: The set of hyperexponential solutions of $L$

1. Compute the singular points $\xi_0, \ldots, \xi_d \in \mathbb{K} \cup \{\infty\}$ of $L$ and the local exponential parts $E_{i,0}, \ldots, E_{i,r_i}$ at each $\xi_i$
2. For each tuple $\mathbf{u} = (u_0, \ldots, u_d)$ with $u_i \leqslant r_i$
   a. Let $e_{\mathbf{u}}(x)$ be a representative of the global exponential part $E_{0,u_0} \cdots E_{d,u_d}$
   b. Write $y(x) = e_u(x)\, \tilde{y}(x)$ in $L \cdot y = 0$; compute an operator $L_{\mathbf{u}}$ annihilating $\tilde{y}$
   c. Compute the space $V_{\mathbf{u}}$ of rational solutions of $L_{\mathbf{u}}$
3. Return $\bigcup_{\mathbf{u}} e_{\mathbf{u}}(x)\, V_{\mathbf{u}}$.

# Final remarks



- Combinatorial explosion: $L$ of order $r$ and deg $d$

  Up to $d+1$ singular points $\xi_i$ $\Rightarrow$ up to $r^{d+1}$ tuples $\mathbf{u}$
  $r$ local exponential parts at each $\xi_i$

- There is a faster algorithm that avoids this explosion [van Hoeij 1997]

- A hyperexponential solution of $L$ gives a first-order right-hand factor.

  Then divide and continue looking for solutions!

- Right-hand factors of arbitrary order
  reduce to first-order factors of auxiliary equations