Computing terms of P-finite sequences

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Doctoral funding available

Group	MAX team, École polytechnique
Starting date	Fall 2025 (negotiable)
Topic	Differential equations From computational complexity and differential Galois theory to low-level implementation details depening on student interests
Advisors	J. van der Hoeven + one of { G. Lecerf, M. Mezzarobba, F. Ollivier, G. Pogudin }

Talk to us ASAP if interested — Tell your friends

1 Introduction

Reminders: D-finite series, P-finite sequences

Theorem. Let $f = \sum_{n \geqslant 0} f_n x^n \in \mathbb{K}[[x]]$.

f is **D-finite** $\Leftrightarrow$ $(f_n)_{n \in \mathbb{N}}$ is **P-finite** / **P-recursive**

$\Leftrightarrow$ f satisfies a linear ODE with coefficients in $\mathbb{K}[x]$

$\Leftrightarrow$ (f_n) satisfies a linear recurrence with coefficients in $\mathbb{K}[n]$

$\Leftrightarrow$ $\dim \operatorname{span}_{\mathbb{K}(x)}(f, f', f'', \dots) < \infty$

Corollary. One can compute the first N terms of a D-finite series in $O(N)$ ops.

"ops" = operations in the base field $\mathbb{K}$

Reminders: C-finite sequences

Definition. A sequence $(u_n) \in \mathbb{K}^{\mathbb{N}}$ is called **C-finite** when it satifies a linear recurrence

$$\forall n \in \mathbb{N}, \quad 1\, u_{n+s} + c_{s-1}\, u_{n+s-1} + \cdots + c_0\, u_n = 0 \qquad \text{with } c_i \in \mathbb{K}.$$

Theorem. One can compute the Nth term of a C-finite sequence

- in $O\big(s^\theta \log(N)\big)$ ops by binary powering on the companion matrix,
- in $O\big(M(s) \log(N)\big)$ ops by binary powering modulo charpoly *or* by repeated Gräffe transforms.

Remarks.

- Over $\mathbb{Z}$, all three methods take $O\big(M_{\mathbb{Z}}(N)\big)$ **bit** operations.
- They do not work in the P-finite case.

Reminders: Binary splitting for hypergeometric sums

Definition. A **(generalized) hypergeometric series** is a power series whose coefficient sequence satisfies a first-order recurrence relation with polynomial coefficients:

$$f(x) = \sum_{n=0}^{\infty} u_n x^n \quad \text{where} \quad u_{n+1} = \frac{p(n)}{q(n)} u_n, \quad u_0 = 1.$$

For $p, q \in \mathbb{Z}[n]$ and $x \in \mathbb{Q}$, one can compute $\sum_{n=0}^{N-1} u_n x^n$ in $O(M_{\mathbb{Z}}(N \log(N)^2))$ **bit** operations

by splitting $\sum_0^{N-1}$ as $\sum_0^{m-1} + \sum_m^{N-1} = \frac{T(0,m)}{Q(0,m)} + \frac{T(m,N)}{Q(m,N)} u_m$

and computing the numerators & denominators recursively

C-Finite sequences: The direct algorithm over $\mathbb{Z}$

$$u_{n+s} + c_{s-1} u_{n+s-1} + \cdots + c_0 u_n = 0$$

$$u_n = \alpha_1^n p_1(n) + \cdots + \alpha_k^n p_k(n)$$

Direct algorithm:
$$\begin{cases} u_s & := & -(c_{s-1} u_{s-1} + \cdots + c_0 u_0) \\ u_{s+1} & := & -(c_{s-1} u_s + \cdots + c_0 u_1) \\ \vdots & & \end{cases}$$

$$|u_n| \leqslant 2^{Kn}$$

Output size can reach $\Omega(N^2)$ for N terms
$\Omega(N)$ for one term

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Bit operations: $\sum_{n=s}^{N-1} C\, M_{\mathbb{Z}}(h, K\, n)$ for a fixed rec.

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C-finite sequences: Binary powering over $\mathbb{Z}$

Proposition. Let $(u_n)_{n\in\mathbb{N}}$ satisfy a linear recurrence with constant coefficients and unit leading term. Assume $u_0 = 1$.

Given $N \in \mathbb{N}$, one can compute u_N in $O(M_{\mathbb{Z}}(N))$ bit operations.

Proof. Write

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+s} \end{pmatrix} = \underbrace{\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -c_0 & -c_1 & \cdots & -c_{s-1} \end{pmatrix}}_{A\in\mathbb{Z}^{s\times s}} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+s-1} \end{pmatrix}.$$

- $\|A^n\| \leqslant \|A\|^n \leqslant 2^{Kn}$ for some $K > 0$.
- Cost of binary powering:

$$C \cdot M_{\mathbb{Z}}(K) + \cdots + C \cdot M_{\mathbb{Z}}\left(\frac{N}{4}K\right) + C \cdot M_{\mathbb{Z}}\left(\frac{N}{2}K\right) = O(M_{\mathbb{Z}}(N)). \qquad \square$$

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Exercise. What is the complexity over $\mathbb{Q}$ (i.e. for a non-unit leading term)?

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$$\text{size}(n!) = 1 + \lfloor \log_2(n!) \rfloor = n \log_2 n + O(n) \qquad \text{(Stirling)}$$

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  Step n is a multiplication of

$$n \log_2 n + O(n) \quad \text{by} \quad \log_2 n + O(1) \quad \text{bits}$$

  costing $n\, M_{\mathbb{Z}}(\log_2 n) + O(n)$ bit operations if done by blocks.

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Quasi-optimal for N terms, unsatisfactory for a single term

Nonsingular recurrences

Definition. We will say that the recurrence relation

$$b_s(n)\, u_{n+s} + \cdots + b_1(n)\, u_{n+1} + b_0(n)\, u_n = 0 \tag{Rec}$$

is **nonsingular** if $b_s(n) \neq 0$ for all $n \in \mathbb{N}$.

Proposition. If (Rec) is nonsingular, then

- its solution space has dimension s,
- any solution $(u_n)_{n\in\mathbb{N}}$ is determined by $(u_0, \dots, u_{s-1})$.

In other words: there is a basis of solutions of the form

$$\begin{aligned} u^{(0)} &= (1, 0, 0, \dots, 0, *, *, *, \dots) \\ u^{(1)} &= (0, 1, 0, \dots, 0, *, *, *, \dots) \\ &\vdots \\ u^{(s-1)} &= (0, 0, 0, \dots, 1, *, *, *, \dots) \end{aligned}$$

(We will study singular recurrences in the next lecture.)

First N terms, Nth term

$$b_s(n)\,u_{n+s} + \cdots + b_1(n)\,u_{n+1} + b_0(n)\,u_n = 0$$

Problems. Given a nonsingular recurrence as above, initial values $u_{0:s}$, and $N \in \mathbb{N}$:

a) Compute $(u_0, \ldots, u_{N-1})$

b) Compute u_N

Complexity models: operations in $\mathbb{K}$ ("ops")
binary operations for $\mathbb{K} = \mathbb{Z}$

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Direct algorithm: repeat $u_n = -\frac{1}{b_s(n-s)}\left(b_{s-1}(n-s)\,u_{n-1} + \cdots + b_0(n-s)\,u_{n-s}\right)$

Arithmetic cost:
Over $\mathbb{Z}$ with $b_s = 1$:

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Over $\mathbb{Z}$ with $b_s = 1$: $O(N^2\,M_{\mathbb{Z}}(\log N))$ binops

Quasi-optimal (for a fixed rec.) for problem a) $\longrightarrow$ Focus on problem b)

2 Baby steps, giant steps

A baby steps-giant steps algorithm for N!

[Strassen 1976]

$$N! = \underbrace{\underbrace{1 \cdot 2 \cdots \ell}\cdot\underbrace{(\ell+1)(\ell+2)\cdots(2\ell)}\cdots\underbrace{(\ell^2-\ell+1)(\ell^2-\ell+2)\cdots\ell^2}}_{N^{1/2}\text{ blocks of size } N^{1/2}} \qquad \ell = N^{1/2}$$

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Total $O(M(N^{1/2})\log N)$ ops

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[Strassen 1976]

Idea: if N is composite, $\lfloor\sqrt{N}\rfloor! \wedge N$ is a nontrivial factor

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Total $O\big(M(N^{1/4})\log(N)^{2+o(1)}\big)$

A TIME-SPACE TRADEOFF FOR LEHMAN'S DETERMINISTIC INTEGER FACTORIZATION METHOD

MARKUS HITTMEIR

ABSTRACT. Fermat's well-known factorization algorithm is based on finding a representation of natural numbers N as the difference of squares. In 1895, Lawrence generalized this idea and applied it to multiples kN of the original number. A systematic approach to choose suitable values for k has been introduced by Lehman in 1974, which resulted in the first deterministic factorization algorithm considerably faster than trial division. In this paper, we construct a time-space tradeoff for Lawrence's generalization and apply it together with Lehman's result to obtain a deterministic integer factorization algorithm with runtime complexity $O(N^{2/9+o(1)})$. This is the first exponential improvement since the establishment of the $O(N^{1/4+o(1)})$ bound in 1977.

1. Introduction

We consider the problem of computing the prime factorization of natural numbers N. There is a large variety of probabilistic and heuristic factorization methods achieving subexponential complexity. We refer the reader to the survey [Len00] and to the monographs [Rie94] and [Wag13]. The focus of the present paper is a more theoretical aspect of the integer factorization problem, which concerns *deterministic* algorithms

AN EXPONENT ONE-FIFTH ALGORITHM FOR DETERMINISTIC INTEGER FACTORISATION

DAVID HARVEY

Abstract. Hittmeir recently presented a deterministic algorithm that provably computes the prime factorisation of a positive integer N in $N^{2/9+o(1)}$ bit operations. Prior to this breakthrough, the best known complexity bound for this problem was $N^{1/4+o(1)}$, a result going back to the 1970s. In this paper we push Hittmeir's techniques further, obtaining a rigorous, deterministic factoring algorithm with complexity $N^{1/5+o(1)}$.

1. Introduction

Let $\mathsf{F}(N)$ denote the time required to compute the prime factorisation of an integer $N \geqslant 2$. By "time" we mean "number of bit operations", or more precisely, the number of steps performed by a deterministic Turing machine with a fixed, finite number of linear tapes [Pap94]. All integers are assumed to be encoded in the usual binary representation.

In this paper we prove the following result:

Theorem 1.1. *There is an integer factorisation algorithm achieving*

$$\mathsf{F}(N) = O(N^{1/5} \log^{16/5} N).$$

Generalization to P-recursive sequences

[Chudnovsky & Chudnovsky 1987]

Write the recurrence in matrix form, pull out the denominator:

$$\begin{pmatrix} u_{n+1} \\ \vdots \\ u_{n+s-1} \\ u_{n+s} \end{pmatrix} = \frac{1}{b_s(n)} \underbrace{\begin{pmatrix} b_s(n) & & & \\ & & \ddots & \\ & & & b_s(n) \\ -b_0(n) & -b_1(n) & \cdots & -b_{s-1}(n) \end{pmatrix}}_{B(n)} \underbrace{\begin{pmatrix} u_n \\ \vdots \\ u_{n+s-2} \\ u_{n+s-1} \end{pmatrix}}_{U_n}$$

Then

$$U_N = \frac{1}{b_s(N-1) \cdots b_s(1)\, b_s(0)} B(N-1) \cdots B(1)\, B(0)\, U_0$$

$B(n) =$ matrix of polynomials of degree $< d$

Fast polynomial matrix "factorial"

Algorithm. *Input*: $B \in \mathbb{K}[n]^{s \times s}$ of deg $< d$, $N \in \mathbb{N}$ *Output*: $B(N-1) \cdots B(1)\, B(0)$

1. Write $N = \ell\, m$ with $\ell =$ and $m =$ (assumed exact for simplicity)
2. Baby steps:
   a. Compute $B(X+1), \ldots, B(X+\ell-1)$
   b. Compute $F(X) = B(X+\ell-1) \cdots B(X+1)\, B(X)$
3. Giant steps:
   a. Compute $F(0), F(\ell), \ldots, F((m-1)\,\ell)$ simultaneously
   b. Deduce and return the product $F((m-1)\,\ell) \cdots F(\ell)\, F(0)$

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naïvely step 2a takes $O(\ell\, d^2\, s^2)$ ops

Exercise 7. Design an algorithm to compute $B(x+a)$ from $B(x)$ in $O(M(d) \log d)$ ops.

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$\deg F(X) < \ell\, d$ $\ell\, d \leqslant m$ Total $O\left(M(m) \log(m)\, s^\theta\right)$

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Exercise 8. Design an algorithm to compute $B(x+a)$ from $B(x)$ in $O(M(d) \log d)$ ops.

Nth term of a P-recursive sequence

Algorithm. *Notation as before.*

1. Compute $B(N-1)\cdots B(1)\,B(0)$ by the previous algorithm $\quad O(M(m)\log(m)\,s^{\theta})$
2. Compute $b_s(N-1)\cdots b_s(1)\,b_s(0)$ by the previous algorithm $\quad O(M(m)\log(m))$
3. Divide, return $\quad O(s^2)$

Theorem. Let $(u^{(0)},\ldots,u^{(s-1)})$ be the basis of solutions s.t. $u_i^{(j)}=\delta_{i,j}$ of a nonsingular recurrence of order s and degree $<d$. One can compute the matrix $\left(u_{N+i}^{(j)}\right)_{i,j}\in\mathbb{K}^{s\times s}$ in

$$O\Big(M(\sqrt{N\,d})\log(N\,d)\,s^{\theta}\Big)\quad \text{ops.}$$

Corollary. One can compute the Nth term of a P-recursive sequence given by a nonsingular recurrence in $O(M(\sqrt{N})\log N)$ ops.

3 Binary splitting

Computing $N!$ in quasi-linear time

Computing N! in quasi-linear time

Algorithm. Use a product tree. That is, split the product as

$$N! = \underbrace{1 \cdot 2 \cdots m}_{P(0,m)} \cdot \underbrace{(m+1) \cdots N}_{P(m,N)}, \qquad m = \lfloor N/2 \rfloor,$$

and recurse.

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and recurse.

Using $\mathrm{size}(P(\ell,h)) \leqslant 1 + (h-\ell)\log_2 N$, the cost $C(\ell,h)$ of computing $P(\ell,h)$ satisfies

$$C(\ell,h) \leqslant C(\ell,m) + C(m,h) + M_{\mathbb{Z}}\big(1 + \lceil (h-\ell)/2 \rceil \log_2 N\big) \qquad m = \lfloor (\ell+h)/2 \rfloor.$$

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The total cost of the multiplications at any given recursion depth is

$$\leqslant \sum_i M_{\mathbb{Z}}\big(1 + \lceil H_i/2 \rceil \log_2 N\big) \qquad \text{where} \quad \sum_i H_i \leqslant N$$

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Total $O\big(M_{\mathbb{Z}}(N \log N) \log N\big)$.

Nth term of a P-recursive sequence

[Chudnovsky & Chudnovsky 1987]

$$b_s(n)\, u_{n+s} + \cdots + b_1(n)\, u_{n+1} + b_0(n)\, u_n = 0, \quad b_i \in \mathbb{Z}[n]$$

Same idea as before:

write $U_n = (u_n, \ldots, u_{n+s-1})$ and $U_N = \dfrac{1}{b_s(N-1) \cdots b_s(1)\, b_s(0)} B(N-1) \cdots B(1)\, B(0)\, U_0$

Algorithm.

1. Compute $B(N-1) \cdots B(1)\, B(0)$ by **binary splitting**
2. Compute $b_s(N-1) \cdots b_s(1)\, b_s(0)$ by **binary splitting**
3. Divide

Theorem. One can compute the Nth term of a sequence $(u_n) \in \mathbb{Q}^{\mathbb{N}}$ given by a nonsingular recurrence with coefficients in $\mathbb{Z}[n]$ in bit operations.

Nth term of a P-recursive sequence

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Algorithm. (costs for fixed recurrence, hides dependency on s and d)

1. Compute $B(N-1)\cdots B(1)\,B(0)$ by **binary splitting** $O(M(n\log n)\log(n))$
2. Compute $b_s(N-1)\cdots b_s(1)\,b_s(0)$ by **binary splitting** $O(M(n\log n)\log(n))$
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3. Divide (gcd!) $O(M(n\log n)\log(n))$

Theorem. One can compute the Nth term of a sequence $(u_n)\in\mathbb{Q}^{\mathbb{N}}$ given by a nonsingular recurrence with coefficients in $\mathbb{Z}[n]$ in $O(M(n\log^2 n))$ bit operations.

An application

[Flajolet & Salvy 1997]

Problem. Compute the coefficient of x^{2N} in

$$(1+x)^{2N}(1+x+x^2)^N.$$

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Let $f(x) = (1+x)^{2N}(1+x+x^2)^N$. One has

$$\frac{f'(x)}{f(x)} = 2N\frac{1}{1+x} + N\frac{2x+1}{1+x+x^2}.$$

Convert ODE to recurrence, use binary splitting.

The case of hypergeometric sums

Goal. Compute $\Sigma_N = \sum_{n=0}^{N-1} u_n$ where $u_{n+1} = \frac{p(n)}{q(n)} u_n, \quad u_0 = 1$

Last week's version.

Write $\sum_{n=\ell}^{h-1} u_n = \sum_{n=\ell}^{h-1} \frac{p(n-1)\cdots p(\ell)}{q(n-1)\cdots q(\ell)} u_\ell = \frac{T(\ell,h)}{Q(\ell,h)} u_\ell$ where $Q(\ell,h) = q(h-1)\cdots q(\ell)$

$$u_h = \frac{P(\ell,h)}{Q(\ell,h)} u_\ell \qquad P(\ell,h) = p(h-1)\cdots p(\ell)$$

Then $\sum_{n=\ell}^{h-1} u_n = \sum_{n=\ell}^{m-1} u_n + \sum_{n=m}^{h-1} u_n$ gives $\frac{T(\ell,h)}{Q(\ell,h)} u_\ell = \frac{T(\ell,m)}{Q(\ell,m)} u_\ell + \frac{T(m,h)}{Q(m,h)} \frac{P(\ell,m)}{Q(\ell,m)} u_\ell$

$$T(\ell,h) = Q(m,h)\, T(\ell,m) + P(\ell,m)\, T(m,h).$$

Matrix version.

$$\begin{pmatrix} u_{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{q(n)} \underbrace{\begin{pmatrix} p(n) & 0 \\ q(n) & q(n) \end{pmatrix}}_{B(n)} \begin{pmatrix} u_n \\ \Sigma_n \end{pmatrix}, \qquad B(h-1)\cdots B(\ell) = \begin{pmatrix} P(\ell,h) & 0 \\ T(\ell,h) & Q(\ell,h) \end{pmatrix}$$

An exercise for next time

Exercise. Give an algorithm to convert an n-bit number from base 2 to base 10 in $O(M_{\mathbb{Z}}(n) \log n)$ bit operations, where $M_{\mathbb{Z}}(n)$ is a bound on the cost of n-bit integer multiplication.

4 Partial sums of D-finite series

Application to sums of D-finite series

Let $\Sigma_n = \sum_{k=0}^{n-1} u_k \xi^k$ for some fixed $\xi \in \mathbb{R}$.

If $(u_n)_{n\in\mathbb{N}}$ satisfies a rec. with poly. coeffs, then (Σ_n) too. (why?)

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Better formulation:

$$\begin{pmatrix} u_{n+1}\,\xi^{n+1} \\ \vdots \\ u_{n+s}\,\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & & \\ & B(n)\,\xi & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \quad 0 \quad \cdots \quad 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n\,\xi^n \\ \vdots \\ u_{n+s-1}\,\xi^n \\ \Sigma_n \end{pmatrix}$$

BSGS evaluation of D-finite series (sketch)

$$\begin{pmatrix} u_{n+1}\,\xi^{n+1} \\ \vdots \\ u_{n+s}\,\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & & \\ & B(n)\,\xi & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \quad 0 \quad \cdots \quad 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n\,\xi^n \\ \vdots \\ u_{n+s-1}\,\xi^n \\ \Sigma_n \end{pmatrix}$$

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Working with p-bit approximations and ignoring rounding errors:

$$\Sigma_N \text{ to } p\text{-bit precision} \quad \text{in} \quad O\big(M(\sqrt{N})\log(N)\,M_{\mathbb{Z}}(p)\big) \text{ ops}$$

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$$\Sigma_N \text{ to } p\text{-bit precision} \quad \text{in} \quad O\big(M(\sqrt{N})\log(N)\,M_{\mathbb{Z}}(p)\big) \text{ ops}$$

Target accuracy 2^{-t} typically requires $N = O(t)$ (geometric convergence)

If rounding errors negligible, working precision $p = t + O(1)$

$\rightsquigarrow$ evaluation of D-finite series to precision t in $\tilde{O}(t^{3/2})$ ops

Binary splitting for D-finite series

Again: $\Sigma_n = \sum_{k=0}^{n-1} u_k \xi^k$ satisfies a recurrence

(Note that ξ enters into the recurrence!)

The previous result on binary splitting yields:

Corollary. One can evaluate the Nth partial sum of a fixed D-finite series at a fixed point $\xi \in \mathbb{Q}$ in $O(M(N \log^2 N))$ bit operations.

Typical case: $N = O(t)$

$t =$ target bit accuracy

Application: High-precision evaluation of classical constants

- $e = \exp(1)$ with error $\leqslant 2^{-t}$ in $O\big(M(t \log t)\big)$ bit operations

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$$e = \sum_{k=0}^{n-1} \frac{1}{k!} + \underbrace{\frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{(n+1)\cdots(n+k)}}_{\leqslant e/n!}, \qquad \frac{e}{n!} \leqslant 2^{-t} \text{ for } n = \frac{t + o(t)}{\log_2 t}$$

Application: High-precision evaluation of classical constants

- $e = \exp(1)$ with error $\leqslant 2^{-t}$ in $O\big(M(t \log t)\big)$ bit operations

$$e = \sum_{k=0}^{n-1} \frac{1}{k!} + \underbrace{\frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{(n+1)\cdots(n+k)}}_{\leqslant e/n!}, \qquad \frac{e}{n!} \leqslant 2^{-t} \text{ for } n = \frac{t + o(t)}{\log_2 t}$$

Cost of the binary splitting method: $O\left(M\left(\frac{t}{\log_2 t} \log\left(\frac{t}{\log_2 t}\right)^2\right)\right) = O(M(t \log t))$.

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- $$\frac{1}{\pi} = \frac{12}{c^{3/2}} \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(3n)!\, n!^3} \frac{(a\, n + b)}{c^{3n}}, \qquad \begin{cases} a = 545140134 \\ b = 13591409 \\ c = 640320 \end{cases}$$ [Chudnovsky² 1987]

1 hypergeometric series, 1 square root, 1 division

Used in record computations — although another algo. yields t digits of π in only $O(M(t) \log t)$ bit ops [Salamin 1976, Brent 1978]

Dependency on the evaluation point

$$\begin{pmatrix} u_{n+1}\,\xi^{n+1} \\ \vdots \\ u_{n+s}\,\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & & \\ & B(n)\,\xi & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \;\; 0 \;\; \cdots \;\; 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n\,\xi^n \\ \vdots \\ u_{n+s-1}\,\xi^n \\ \Sigma_n \end{pmatrix}$$

If ξ is of bit size h, then (for a fixed differential equation):

- the matrices, taken at $n \leqslant N$, have bit size $O(h + \log N)$,
- the cost of computing the product tree for N terms becomes

$$O\Big(M\big(\overbrace{N\,\underbrace{(h + \log N)}_{\text{size of each leaf}}}^{\text{size of each row}}\big)\underbrace{\log N}_{\text{depth}}\Big).$$

If N and h are both $\Theta(t)$, the cost becomes quadratic in t!

5 The “bit-burst” method

Fast high-precision evaluation of the exponential function

[Brent 1976]

Goal: for a real number $\frac{1}{2} \leqslant \xi < 1$, compute $\exp(\xi)$ with error $\leqslant 2^{-t}$ in $\tilde{O}(t)$ bit ops.

We assume that a sufficiently accurate approximation of ξ is given ($t + O(1)$ bits suffice)

Remark: can reduce to $\xi \in [1/2, 1)$ using $\exp(2x) = \exp(x)^2$.

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$$\text{Write} \qquad \xi = 0.\underbrace{\xi_1}\underbrace{\xi_2\xi_3}\underbrace{\xi_4\xi_5\xi_6\xi_7}\underbrace{\xi_8\xi_9\xi_{10}\xi_{11}\xi_{12}\xi_{13}\xi_{14}\xi_{15}}\xi_{16}\xi_{17}\ldots$$

$$= m_0 + m_1 + m_2 + \cdots + m_{K-1} \qquad \text{where } \begin{cases} m_k \leqslant 2^{-2^k+1} \\ m_k \text{ fits on } 2^k \text{ bits} \end{cases}$$

$$\text{Then} \qquad \exp(\xi) = \exp(m_0)\exp(m_1)\cdots\exp(m_{K-1}) \qquad \text{and } K = O(\log t)$$

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Algorithm. Evaluate each m_k by binary splitting, then multiply.

The final multiplications cost $O(M(t)\log t)$ in total.

Remark: can reduce to $\xi \in [1/2, 1)$ using $\exp(2x) = \exp(x)^2$.

Fast high-precision exponential: analysis

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$$O\Big(M\big(\overbrace{N\ \underbrace{(h + \log N)}_{\text{size of each leaf}}}^{\text{size of each row}}\big)\ \underbrace{\log N}_{\text{depth}}\Big)$$

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Total: $\sum_{k=0}^{K-1} C\, M(t \log t + 2^{-k} t \log^2 t)$

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The bit-burst method for D-finite series

[Chudnovsky & Chudnovsky 1987]

Fix a differential operator L; assume that 0 is an ordinary point.

Consider a basis $y_1, \ldots, y_r$ of analytic solutions.

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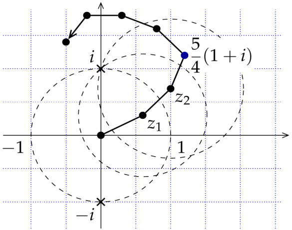
- By multiplying these matrices for steps corresponding to a decomposition

$$\xi = 0.\underbrace{\xi_1}\underbrace{\xi_2\xi_3}\underbrace{\xi_4\xi_5\xi_6\xi_7}\underbrace{\xi_8\xi_9\xi_{10}\xi_{11}\xi_{12}\xi_{13}\xi_{14}\xi_{15}}\xi_{16}\xi_{17}\ldots$$

  we can evaluate the solutions at complex points of bit size t in $\tilde{O}(t)$ ops.

Fast high-precision evaluation of D-finite functions (sketch)

[Chudnovsky & Chudnovsky 1987, van der Hoeven 1999, …]



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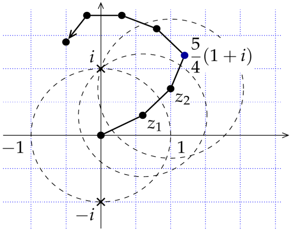

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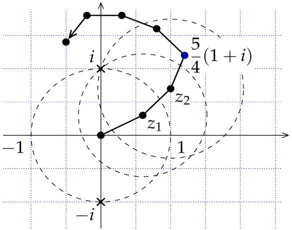

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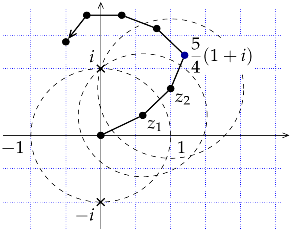

- By multiplying matrices $\begin{pmatrix} y_1(\xi) & \cdots & y_r(\xi) \\ \vdots & & \vdots \\ y_1^{(r-1)}(\xi) & \cdots & y_r^{(r-1)}(\xi) \end{pmatrix}$, we can also evaluate the (analytic continuation of) the solutions outside the disk $|\xi| < \rho$.
- For fixed ξ, computing $y(\xi)$ with an error $\leqslant 2^{-t}$ requires $O(t)$ digits of ξ.
- All necessary error bounds can be computed automatically.

Pseudo-theorem: “one can evaluate a **fixed** D-finite function at a **fixed** point $\in \mathbb{C}$ with an error $\leqslant 2^{-t}$ in $\tilde{O}(t)$ bit operations”.

(Can be stated rigorously with more care.)

6 Rectangular splitting

Rectangular splitting for polynomials

[Paterson & Stockmeyer 1973]

Goal: evaluate $p(\xi) = a_{d-1}\,\xi^{d-1} + \cdots + a_0$ with "small" a_i at a "large" (p-bit) $\xi \in \mathbb{R}$

$$\begin{aligned} p(x) = {} & (a_0 + a_1 x + \cdots + a_{\ell-1} x^{\ell-1})\, x^0 \qquad (d = m\,\ell) \\ + {} & (a_\ell + a_{\ell+1} x + \cdots + a_{2\ell-1} x^{\ell-1})\, x^\ell \\ & \vdots \\ + {} & (a_{(m-1)\ell} + a_{(m-1)\ell+1} x + \cdots + a_{m\ell-1} x^{\ell-1})\, x^{(m-1)\ell} \end{aligned}$$

Same idea for evaluating $p \in \mathbb{K}[x]$ on a polynomial / matrix / …

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$$\begin{array}{rcllllllll} p(x) & = & & (& a_0 & + & a_1\,x & + \cdots + & a_{\ell-1}\,x^{\ell-1} &)\; x^0 \qquad (d = m\,\ell) \\ & & + & (& a_\ell & + & a_{\ell+1}\,x & + \cdots + & a_{2\ell-1}\,x^{\ell-1} &)\; x^\ell \\ & & \vdots & & & & & & & \\ & & + & (& a_{(m-1)\ell} & + & a_{(m-1)\ell+1}\,x & + \cdots + & a_{m\ell-1}\,x^{\ell-1} &)\; x^{(m-1)\ell} \end{array}$$

Algorithm.

1. (Baby steps) Compute $\xi^2, \ldots, \xi^\ell$ — $O(\ell)$ costly ops
2. Evalute the inner polynomials — $O(\ell\,m)$ cheap ops
3. (Giant steps) Compute $\xi^{2\ell}, \ldots, \xi^{(m-1)\ell}$ — $O(m)$ costly ops
4. Evaluate the outer polynomial — $O(m)$ costly ops

Same idea for evaluating $p \in \mathbb{K}[x]$ on a polynomial / matrix / ...

Rectangular splitting for hypergeometric series

$$f(x) = a_0 + a_0\,a_1\,x + a_0\,a_1\,a_2\,x^2 + \cdots \qquad a_n = p(n)\,/\,q(n)$$

$$\begin{aligned}
& && a_0\,(1 + a_1\,(x + a_2\,(x^2 + \cdots + a_{\ell-1}\,x^{\ell-1})))\;x^0 \\
+ & \quad a_0 \cdots a_{\ell-1} && \Big(\;a_\ell\,(1 + a_{\ell+1}\,(x + a_{\ell+2}\,(x^2 + \cdots + a_{2\ell-1}\,x^{\ell-1})))\;x^\ell \\
+ & \quad a_\ell \cdots a_{2\ell-1} && \Big(\qquad\qquad \ldots \\
+ & \quad a_{(m-1)\ell} \cdots a_{m\ell-1} && \Big(\;a_{(m-1)\ell}\,(1 + \cdots(\cdots(\;\cdots\; + a_{m\ell-1}\,x^{\ell-1})))\;x^{(m-1)\ell}\Big)\ldots\Big)\Big)
\end{aligned}$$