Computing terms of P-finite sequences

Marc Mezzarobba

January 27, 2025

Doctoral funding available

Group	MAX team, École polytechnique
Starting date	Fall 2025 (negotiable)
Topic	Differential equations From computational complexity and differential Galois theory to low-level implementation details depening on student interests
Advisors	J. van der Hoeven + one of { G. Lecerf, M. Mezzarobba, F. Ollivier, G. Pogudin }

Talk to us ASAP if interested — Tell your friends

1 Introduction

Reminders: D-finite series, P-finite sequences

Theorem. Let $f = \sum_{n \geqslant 0} f_n x^n \in \mathbb{K}[[x]]$.

f is **D-finite**	⇔ $(f_n)_{n\in\mathbb{N}}$ is **P-finite** / **P-recursive**
⇔ f satisfies a linear ODE with coefficients in $\mathbb{K}[x]$	⇔ (f_n) satisfies a linear recurrence with coefficients in $\mathbb{K}[n]$
⇔ $\dim \operatorname{span}_{\mathbb{K}(x)}(f, f', f'', \dots) < \infty$	

Corollary. One can compute the first N terms of a D-finite series in $O(N)$ ops.

"ops" = operations in the base field $\mathbb{K}$

Reminders: C-finite sequences

Definition. A sequence $(u_n) \in \mathbb{K}^{\mathbb{N}}$ is called **C-finite** when it satisfies a linear recurrence

$$\forall n \in \mathbb{N}, \quad 1\, u_{n+s} + c_{s-1} u_{n+s-1} + \cdots + c_0 u_n = 0 \qquad \text{with } c_i \in \mathbb{K}.$$

Theorem. One can compute the Nth term of a C-finite sequence

- in $O(s^\theta \log(N))$ ops by binary powering on the companion matrix,
- in $O(M(s)\log(N))$ ops by binary powering modulo charpoly *or* by repeated Gräffe transforms.

Remarks.

- Over $\mathbb{Z}$, all three methods take $O(M_{\mathbb{Z}}(N))$ **bit** operations.
- They do not work in the P-finite case.

Reminders: Binary splitting for hypergeometric sums

Definition. A **(generalized) hypergeometric series** is a power series whose coefficient sequence satisfies a first-order recurrence relation with polynomial coefficients:

$$f(x) = \sum_{n=0}^{\infty} u_n x^n \quad \text{where} \quad u_{n+1} = \frac{p(n)}{q(n)} u_n, \quad u_0 = 1.$$

For $p, q \in \mathbb{Z}[n]$ and $x \in \mathbb{Q}$, one can compute $\sum_{n=0}^{N-1} u_n x^n$ in $O(M_{\mathbb{Z}}(N \log(N)^2))$ **bit** operations

by splitting $\sum_0^{N-1}$ as $\sum_0^{m-1} + \sum_m^{N-1} = \frac{T(0,m)}{Q(0,m)} + \frac{T(m,N)}{Q(m,N)} u_m$ and computing the numerators & denominators recursively

C-Finite sequences: The direct algorithm over $\mathbb{Z}$

$$u_{n+s} + c_{s-1} u_{n+s-1} + \cdots + c_0 u_n = 0$$

$$u_n = \alpha_1^n p_1(n) + \cdots + \alpha_k^n p_k(n)$$

$$|u_n| \leqslant 2^{Kn}$$

Direct algorithm: $\begin{cases} u_s := -(c_{s-1} u_{s-1} + \cdots + c_0 u_0) \\ u_{s+1} := -(c_{s-1} u_s + \cdots + c_0 u_1) \\ \vdots \end{cases}$

Bit operations: $\displaystyle\sum_{n=s}^{N-1} C\, M_{\mathbb{Z}}(h, K n) \leqslant C' \frac{N(N-1)}{2} = O(N^2)$ for a fixed rec.

Output size can reach $\Omega(N^2)$ for N terms
$\Omega(N)$ for one term

C-finite sequences: Binary powering over $\mathbb{Z}$

Proposition. Let $(u_n)_{n \in \mathbb{N}}$ satisfy a linear recurrence with constant coefficients and unit leading term. Assume $u_0 = 1$.
Given $N \in \mathbb{N}$, one can compute u_N in $O(M_{\mathbb{Z}}(N))$ bit operations.

Proof. Write

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+s} \end{pmatrix} = \underbrace{\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -c_0 & -c_1 & \cdots & -c_{s-1} \end{pmatrix}}_{A \in \mathbb{Z}^{s \times s}} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+s-1} \end{pmatrix}.$$

- $\|A^n\| \leqslant \|A\|^n \leqslant 2^{Kn}$ for some $K > 0$.
- Cost of binary powering:

$$C \cdot M_{\mathbb{Z}}(K) + \cdots + C \cdot M_{\mathbb{Z}}(\tfrac{N}{4} K) + C \cdot M_{\mathbb{Z}}(\tfrac{N}{2} K) = O(M_{\mathbb{Z}}(N)). \qquad \square$$

Direct computation of N!

Algorithm. Repeat $u_n = n \cdot u_{n-1}$ for $n = 1, 2, \ldots, N$.

- Arithmetic complexity: $O(N)$ ops — Optimal if computing $1!, \ldots, N!$
- Bit complexity:

$$\mathrm{size}(n!) = 1 + \lfloor \log_2(n!) \rfloor = n \log_2 n + O(n) \qquad \text{(Stirling)}$$

  Step n is a multiplication of

$$n \log_2 n + O(n) \quad \text{by} \quad \log_2 n + O(1) \quad \text{bits}$$

  costing $n\, M_{\mathbb{Z}}(\log_2 n) + O(n)$ bit operations if done by blocks.

  Total cost: $$\sum_{n=1}^{N} \left(n\, M_{\mathbb{Z}}(\log_2 n) + O(n)\right) = \frac{N^2}{2} M_{\mathbb{Z}}(\log_2 N) + O(N^2) \text{ bit ops}$$

  Quasi-optimal for N terms, unsatisfactory for a single term

Nonsingular recurrences

Definition. We will say that the recurrence relation

$$b_s(n)\, u_{n+s} + \cdots + b_1(n)\, u_{n+1} + b_0(n)\, u_n = 0 \qquad \text{(Rec)}$$

is **nonsingular** if $b_s(n) \neq 0$ for all $n \in \mathbb{N}$.

Proposition. If (Rec) is nonsingular, then

- its solution space has dimension s,
- any solution $(u_n)_{n \in \mathbb{N}}$ is determined by $(u_0, \ldots, u_{s-1})$.

In other words: there is a basis of solutions of the form

$$\begin{aligned} u^{(0)} &= (1, 0, 0, \ldots, 0, *, *, *, \ldots) \\ u^{(1)} &= (0, 1, 0, \ldots, 0, *, *, *, \ldots) \\ &\vdots \\ u^{(s-1)} &= (0, 0, 0, \ldots, 1, *, *, *, \ldots) \end{aligned}$$

(We will study singular recurrences in the next lecture.)

First N terms, Nth term

$$b_s(n)\, u_{n+s} + \cdots + b_1(n)\, u_{n+1} + b_0(n)\, u_n = 0$$

Problems. Given a nonsingular recurrence as above, initial values $u_{0:s}$, and $N \in \mathbb{N}$:

a) Compute $(u_0, \ldots, u_{N-1})$

b) Compute u_N

Complexity models: operations in $\mathbb{K}$ ("ops")
binary operations for $\mathbb{K} = \mathbb{Z}$

Bit sizes: $O(n \log n)$ for a single u_n, $O(N^2 \log N)$ for $u_{0:N}$ (reached)

Direct algorithm: repeat $u_n = -\frac{1}{b_s(n-s)} \left(b_{s-1}(n-s)\, u_{n-1} + \cdots + b_0(n-s)\, u_{n-s}\right)$

Arithmetic cost: $O(N)$ ops
Over $\mathbb{Z}$ with $b_s = 1$: $O(N^2 M_{\mathbb{Z}}(\log N))$ binops

Quasi-optimal (for a fixed rec.) for problem a) $\longrightarrow$ Focus on problem b)

2 Baby steps, giant steps

A baby steps-giant steps algorithm for N!

[Strassen 1976]

$$N! = \underbrace{\underbrace{1\cdot 2\cdots \ell}\cdot\underbrace{(\ell+1)\,(\ell+2)\cdots(2\,\ell)}\cdots\underbrace{(\ell^2-\ell+1)\,(\ell^2-\ell+2)\cdots\ell^2}}_{N^{1/2}\text{ blocks of size }N^{1/2}} \qquad \ell = N^{1/2}$$

Algorithm. *Input*: N *Output*: $N!$

1. Let $\ell = \lfloor N^{1/2} \rfloor$
2. Baby steps:
   a. Compute $F = (x+1)\,(x+2)\cdots(x+\ell)$ — $O(M(\ell)\log \ell)$
3. Giant steps:
   a. Compute $P_0 = F(0), P_1 = F(\ell), P_2 = F(2\,\ell), \ldots, P_{\ell-1} = F((\ell-1)\,\ell)$ by multipoint evaluation
   b. Return $P_0\,P_1\cdots P_{\ell-1}\cdot(\ell^2+1)\cdots(N-1)\,N$

Deterministic integer factoring

[Strassen 1976]

Idea: if N is composite, $\lfloor\sqrt{N}\rfloor! \wedge N$ is a nontrivial factor

Algorithm. *Input*: N *Output*: a nontrivial factor of N, or 1 if N is prime

1. Let $\ell = \lceil N^{1/4} \rceil$
2. Baby steps:
   a. Compute $F = (x+1)\,(x+2)\cdots(x+\ell) \in (\mathbb{Z}/N\mathbb{Z})[x]$ — $O(M(\ell)\log(\ell)\,M_{\mathbb{Z}}(h))$
3. Giant steps:
   a. Compute $P_0 = F(0), \ldots, P_{\ell-1} = F((\ell-1)\,\ell)$ by mulpt ev. — $O(M(\ell)\log(\ell)\,M_{\mathbb{Z}}(h))$
   b. Compute $P_0 \wedge N, \ldots, P_{\ell-1} \wedge N$ — $O(\ell\,M_{\mathbb{Z}}(h)\log(h))$

$h = 1 + \lfloor \log_2 N \rfloor$ Total $O\big(M(N^{1/4})\log(N)^{2+o(1)}\big)$

A TIME-SPACE TRADEOFF FOR LEHMAN'S DETERMINISTIC INTEGER FACTORIZATION METHOD

MARKUS HITTMEIR

Abstract. Fermat's well-known factorization algorithm is based on finding a representation of natural numbers N as the difference of squares. In 1895, Lawrence generalized this idea and applied it to multiples kN of the original number. A systematic approach to choose suitable values for k has been introduced by Lehman in 1974, which resulted in the first deterministic factorization algorithm considerably faster than trial division. In this paper, we construct a time-space tradeoff for Lawrence's generalization and apply it together with Lehman's result to obtain a deterministic integer factorization algorithm with runtime complexity $O(N^{2/9+o(1)})$. This is the first exponential improvement since the establishment of the $O(N^{1/4+o(1)})$ bound in 1977.

1. Introduction

We consider the problem of computing the prime factorization of natural numbers N. There is a large variety of probabilistic and heuristic factorization methods achieving subexponential complexity. We refer the reader to the survey [Len00] and to the monographs [Rie94] and [Wag13]. The focus of the present paper is a more theoretical aspect of the integer factorization problem, which concerns *deterministic* algorithms

AN EXPONENT ONE-FIFTH ALGORITHM FOR DETERMINISTIC INTEGER FACTORISATION

DAVID HARVEY

Abstract. Hittmeir recently presented a deterministic algorithm that provably computes the prime factorisation of a positive integer N in $N^{2/9+o(1)}$ bit operations. Prior to this breakthrough, the best known complexity bound for this problem was $N^{1/4+o(1)}$, a result going back to the 1970s. In this paper we push Hittmeir's techniques further, obtaining a rigorous, deterministic factoring algorithm with complexity $N^{1/5+o(1)}$.

1. Introduction

Let $\mathsf{F}(N)$ denote the time required to compute the prime factorisation of an integer $N \geqslant 2$. By "time" we mean "number of bit operations", or more precisely, the number of steps performed by a deterministic Turing machine with a fixed, finite number of linear tapes [Pap94]. All integers are assumed to be encoded in the usual binary representation.

In this paper we prove the following result:

Theorem 1.1. *There is an integer factorisation algorithm achieving*

$$\mathsf{F}(N) = O(N^{1/5}\log^{16/5} N).$$

Generalization to P-recursive sequences

[Chudnovsky & Chudnovsky 1987]

Write the recurrence in matrix form, pull out the denominator:

$$\underbrace{\begin{pmatrix} u_{n+1} \\ \vdots \\ u_{n+s-1} \\ u_{n+s} \end{pmatrix}} = \frac{1}{b_s(n)} \underbrace{\begin{pmatrix} & b_s(n) & & \\ & & \ddots & \\ & & & b_s(n) \\ -b_0(n) & -b_1(n) & \cdots & -b_{s-1}(n) \end{pmatrix}}_{B(n)} \underbrace{\begin{pmatrix} u_n \\ \vdots \\ u_{n+s-2} \\ u_{n+s-1} \end{pmatrix}}_{U_n}$$

Then

$$U_N = \frac{1}{b_s(N-1)\cdots b_s(1)\, b_s(0)} B(N-1)\cdots B(1)\, B(0)\, U_0$$

$B(n) =$ matrix of polynomials of degree $< d$

Fast polynomial matrix "factorial"

Algorithm. *Input*: $B \in \mathbb{K}[n]^{s\times s}$ of deg $<d$, $N \in \mathbb{N}$ *Output*: $B(N-1)\cdots B(1)\, B(0)$

1. Write $N = \ell\, m$ with $\ell = (N/d)^{1/2}$ and $m = (N\, d)^{1/2}$ (assumed exact for simplicity)
2. Baby steps:
   a. Compute $B(X+1), \ldots, B(X+\ell-1)$ — $O(\ell\, M(d) \log(d)\, s^2)$
   b. Compute $F(X) = B(X+\ell-1)\cdots B(X+1)\, B(X)$ — $O(M(\ell\, d) \log(\ell)\, s^\theta)$
3. Giant steps:
   a. Compute $F(0), F(\ell), \ldots, F((m-1)\,\ell)$ simultaneously — $O(M(m) \log(m)\, s^\theta)$
   b. Deduce and return the product $F((m-1)\,\ell)\cdots F(\ell)\, F(0)$ — $O(m\, s^\theta)$

$\deg F(X) < \ell\, d$ $\quad \ell\, d \leqslant m$ Total $O\Big(M(m) \log(m)\, s^\theta\Big)$

naïvely step 2a takes $O(\ell\, d^2\, s^2)$ ops

Exercise 1. Design an algorithm to compute $B(x+a)$ from $B(x)$ in $O(M(d) \log d)$ ops.

Nth term of a P-recursive sequence

Algorithm. *Notation as before.*

1. Compute $B(N-1)\cdots B(1)\, B(0)$ by the previous algorithm — $O(M(m) \log(m)\, s^\theta)$
2. Compute $b_s(N-1)\cdots b_s(1)\, b_s(0)$ by the previous algorithm — $O(M(m) \log(m))$
3. Divide, return — $O(s^2)$

Theorem. Let $(u^{(0)}, \ldots, u^{(s-1)})$ be the basis of solutions s.t. $u_i^{(j)} = \delta_{i,j}$ of a nonsingular recurrence of order s and degree $<d$. One can compute the matrix $(u_{N+i}^{(j)})_{i,j} \in \mathbb{K}^{s\times s}$ in

$$O\Big(M(\sqrt{N\, d}) \log(N\, d)\, s^\theta\Big) \quad \text{ops.}$$

Corollary. One can compute the Nth term of a P-recursive sequence given by a nonsingular recurrence in $O(M(\sqrt{N}) \log N)$ ops.

3 Binary splitting

Computing N! in quasi-linear time

Algorithm. Use a product tree. That is, split the product as

$$N! = \underbrace{1 \cdot 2 \cdots m}_{P(0,m)} \cdot \underbrace{(m+1) \cdots N}_{P(m,N)}, \qquad m = \lfloor N/2 \rfloor,$$

and recurse.

Using $\mathrm{size}(P(\ell,h)) \leqslant 1 + (h-\ell)\log_2 N$, the cost $C(\ell,h)$ of computing $P(\ell,h)$ satisfies

$$C(\ell,h) \leqslant C(\ell,m) + C(m,h) + M_{\mathbb{Z}}\big(1 + \lceil (h-\ell)/2 \rceil \log_2 N\big) \qquad m = \lfloor (\ell+h)/2 \rfloor.$$

The total cost of the multiplications at any given recursion depth is

$$\leqslant \sum_i M_{\mathbb{Z}}(1 + \lceil H_i/2 \rceil \log_2 N) \qquad \text{where} \quad \sum_i H_i \leqslant N$$
$$\leqslant M_{\mathbb{Z}}\left(\frac{N}{2}\log_2 N + O(N)\right).$$

Total $O\big(M_{\mathbb{Z}}(N \log N) \log N\big)$.

Nth term of a P-recursive sequence

[Chudnovsky & Chudnovsky 1987]

$$b_s(n)\,u_{n+s} + \cdots + b_1(n)\,u_{n+1} + b_0(n)\,u_n = 0, \quad b_i \in \mathbb{Z}[n]$$

Same idea as before:
write $U_n = (u_n, \ldots, u_{n+s-1})$ and $U_N = \dfrac{1}{b_s(N-1)\cdots b_s(1)\,b_s(0)} B(N-1)\cdots B(1)\,B(0)\,U_0$

Algorithm.

1. Compute $B(N-1)\cdots B(1)\,B(0)$ by **binary splitting**
2. Compute $b_s(N-1)\cdots b_s(1)\,b_s(0)$ by **binary splitting**
3. Divide

Theorem. One can compute the Nth term of a sequence $(u_n) \in \mathbb{Q}^{\mathbb{N}}$ given by a nonsingular recurrence with coefficients in $\mathbb{Z}[n]$ in bit operations.

An application

[Flajolet & Salvy 1997]

Problem. Compute the coefficient of x^{2N} in

$$(1+x)^{2N}(1+x+x^2)^N.$$

Let $f(x) = (1+x)^{2N}(1+x+x^2)^N$. One has

$$\frac{f'(x)}{f(x)} = 2N\frac{1}{1+x} + N\frac{2x+1}{1+x+x^2}.$$

Convert ODE to recurrence, use binary splitting.

The case of hypergeometric sums

Goal. Compute $\Sigma_N = \sum_{n=0}^{N-1} u_n \qquad$ where $\quad u_{n+1} = \dfrac{p(n)}{q(n)} u_n, \quad u_0 = 1$

Last week's version.

Write $\displaystyle\sum_{n=\ell}^{h-1} u_n = \sum_{n=\ell}^{h-1} \frac{p(n-1)\cdots p(\ell)}{q(n-1)\cdots q(\ell)} u_\ell = \frac{T(\ell,h)}{Q(\ell,h)} u_\ell \qquad$ where $Q(\ell,h) = q(h-1)\cdots q(\ell)$

$$u_h = \frac{P(\ell,h)}{Q(\ell,h)} u_\ell \qquad\qquad P(\ell,h) = p(h-1)\cdots p(\ell)$$

Then $\displaystyle\sum_{n=\ell}^{h-1} u_n = \sum_{n=\ell}^{m-1} u_n + \sum_{n=m}^{h-1} u_n \quad$ gives $\quad \dfrac{T(\ell,h)}{Q(\ell,h)} u_\ell = \dfrac{T(\ell,m)}{Q(\ell,m)} u_\ell + \dfrac{T(m,h)}{Q(m,h)}\dfrac{P(\ell,m)}{Q(\ell,m)} u_\ell$

$$T(\ell,h) = Q(m,h)\,T(\ell,m) + P(\ell,m)\,T(m,h).$$

Matrix version.

$$\begin{pmatrix} u_{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \underbrace{\frac{1}{q(n)}\begin{pmatrix} p(n) & 0 \\ q(n) & q(n) \end{pmatrix}}_{B(n)} \begin{pmatrix} u_n \\ \Sigma_n \end{pmatrix}, \qquad B(h-1)\cdots B(\ell) = \begin{pmatrix} P(\ell,h) & 0 \\ T(\ell,h) & Q(\ell,h) \end{pmatrix}$$

An exercise for next time

Exercise. Give an algorithm to convert an n-bit number from base 2 to base 10 in $O(M_{\mathbb{Z}}(n)\log n)$ bit operations, where $M_{\mathbb{Z}}(n)$ is a bound on the cost of n-bit integer multiplication.

4 Partial sums of D-finite series

Application to sums of D-finite series

Let $\Sigma_n = \sum_{k=0}^{n-1} u_k \xi^k$ for some fixed $\xi \in \mathbb{R}$.

If $(u_n)_{n\in\mathbb{N}}$ satisfies a rec. with poly. coeffs, then (Σ_n) too. (why?)

Better formulation:

$$\begin{pmatrix} u_{n+1}\xi^{n+1} \\ \vdots \\ u_{n+s}\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & & \\ & B(n)\,\xi & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \;\; 0 \;\; \cdots \;\; 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n\xi^n \\ \vdots \\ u_{n+s-1}\xi^n \\ \Sigma_n \end{pmatrix}$$

BSGS evaluation of D-finite series (sketch)

$$\begin{pmatrix} u_{n+1}\xi^{n+1} \\ \vdots \\ u_{n+s}\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & & \\ & B(n)\,\xi & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \;\; 0 \;\; \cdots \;\; 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n\xi^n \\ \vdots \\ u_{n+s-1}\xi^n \\ \Sigma_n \end{pmatrix}$$

Working with p-bit approximations and ignoring rounding errors:

Σ_N to p-bit precision in $O\big(M(\sqrt{N})\log(N)\,M_{\mathbb{Z}}(p)\big)$ ops

Target accuracy 2^{-t} typically requires $N = O(t)$ (geometric convergence)

If rounding errors negligible, working precision $p = t + O(1)$

$\rightsquigarrow$ evaluation of D-finite series to precision t in $\tilde{O}(t^{3/2})$ ops

Binary splitting for D-finite series

Again: $\Sigma_n = \sum_{k=0}^{n-1} u_k \xi^k$ satisfies a recurrence

(Note that ξ enters into the recurrence!)

The previous result on binary splitting yields:

Corollary. One can evaluate the Nth partial sum of a fixed D-finite series at a fixed point $\xi \in \mathbb{Q}$ in $O(M(N \log^2 N))$ bit operations.

Typical case: $N = O(t)$

$t =$ target bit accuracy

Application: High-precision evaluation of classical constants

- $e = \exp(1)$ with error $\leqslant 2^{-t}$ in $O(M(t \log t))$ bit operations

  $$e = \sum_{k=0}^{n-1} \frac{1}{k!} + \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \frac{1}{(n+1)\cdots(n+k)}}_{\leqslant e/n!}, \qquad \frac{e}{n!} \leqslant 2^{-t} \text{ for } n = \frac{t + o(t)}{\log_2 t}$$

  Cost of the binary splitting method: $O\left(M\left(\frac{t}{\log_2 t} \log\left(\frac{t}{\log_2 t}\right)^2\right)\right) = O(M(t \log t))$.

- $\ln(2)$ in $O(M(t \log(t)^2))$ bit operations: $\ln(2) = -\ln(1+\xi)$ with $\xi = -\frac{1}{2}$

  Radius of convergence $= 1$ $\Rightarrow$ general term $= O(2^{-k})$ $\Rightarrow$ need $O(t)$ terms.

- $$\frac{1}{\pi} = \frac{12}{c^{3/2}} \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(3n)!\, n!^3} \frac{(a\,n+b)}{c^{3n}}, \qquad \begin{cases} a = 545140134 \\ b = 13591409 \\ c = 640320 \end{cases}$$ [Chudnovsky² 1987]

  1 hypergeometric series, 1 square root, 1 division

  Used in record computations — although another algo. yields t digits of π in only $O(M(t) \log t)$ bit ops [Salamin 1976, Brent 1978]

Dependency on the evaluation point

$$\begin{pmatrix} u_{n+1}\xi^{n+1} \\ \vdots \\ u_{n+s}\xi^{n+1} \\ \Sigma_{n+1} \end{pmatrix} = \frac{1}{b_s(n)} \begin{pmatrix} \begin{pmatrix} & B(n)\,\xi & \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_s(n) \;\; 0 \;\; \cdots \;\; 0 & b_s(n) \end{pmatrix} \begin{pmatrix} u_n \xi^n \\ \vdots \\ u_{n+s-1}\xi^n \\ \Sigma_n \end{pmatrix}$$

If ξ is of bit size h, then (for a fixed differential equation):

- the matrices, taken at $n \leqslant N$, have bit size $O(h + \log N)$,
- the cost of computing the product tree for N terms becomes

$$O\Big(M\big(\overbrace{N\,\underbrace{(h + \log N)}_{\text{size of each leaf}}}^{\text{size of each row}}\big)\underbrace{\log N}_{\text{depth}}\Big).$$

If N and h are both $\Theta(t)$, the cost becomes quadratic in t!

5 The "bit-burst" method

Fast high-precision evaluation of the exponential function

[Brent 1976]

Goal: for a real number $\frac{1}{2} \leqslant \xi < 1$, compute $\exp(\xi)$ with error $\leqslant 2^{-t}$ in $\tilde{O}(t)$ bit ops.

We assume that a sufficiently accurate approximation of ξ is given ($t + O(1)$ bits suffice)

$$\begin{aligned}
\text{Write} \qquad \xi &= 0.\underbrace{\xi_1}\underbrace{\xi_2\xi_3}\underbrace{\xi_4\xi_5\xi_6\xi_7}\underbrace{\xi_8\xi_9\xi_{10}\xi_{11}\xi_{12}\xi_{13}\xi_{14}\xi_{15}}\xi_{16}\xi_{17}\ldots \\
&= m_0 + m_1 + m_2 + \cdots + m_{K-1} \qquad \text{where } \begin{cases} m_k \leqslant 2^{-2^k+1} \\ m_k \text{ fits on } 2^k \text{ bits} \end{cases} \\
\text{Then} \qquad \exp(\xi) &= \exp(m_0)\exp(m_1)\cdots\exp(m_{K-1}) \qquad \text{and } K = O(\log t)
\end{aligned}$$

Algorithm. Evaluate each m_k by binary splitting, then multiply.

The final multiplications cost $O(M(t)\log t)$ in total.

Remark: can reduce to $\xi \in [1/2, 1)$ using $\exp(2x) = \exp(x)^2$.

Fast high-precision exponential: analysis

$$\xi = m_0 + m_1 + m_2 + \cdots + m_{K-1} \qquad \text{where } \begin{cases} m_k \leqslant 2^{-2^k+1} \\ m_k \text{ fits on } 2^k \text{ bits} \end{cases}$$

Computation of a single $\exp(m_k)$:

- Because $m_k \leqslant 2^{-2^k+1}$, only $N = O(2^{-k} t)$ terms of the series are needed
- Cost of binary splitting:

$$\begin{aligned}
O\Big(M\big(\overbrace{N\ \underbrace{(h + \log N)}_{\text{size of each leaf}}}^{\text{size of each row}}\big)\underbrace{\log N}_{\text{depth}}\Big) &= O\Big(M\big(\overbrace{2^{-k} t\ \underbrace{(2^k + \log t)}_{\text{size of each leaf}}}^{\text{size of each row}}\big)\underbrace{\log t}_{\text{depth}}\Big) \\
&= O(M(t\log t + 2^{-k} t \log^2 t))
\end{aligned}$$

Total: $\displaystyle\sum_{k=0}^{K-1} C\, M(t\log t + 2^{-k} t\log^2 t) \leqslant C \cdot M\left(\sum_{k=0}^{K-1} (t\log t + 2^{-k} t \log^2 t)\right) = O(M(t\log(t)^2))$

The bit-burst method for D-finite series

[Chudnovsky & Chudnovsky 1987]

Fix a differential operator L; assume that 0 is an ordinary point.

Consider a basis $y_1, \ldots, y_r$ of analytic solutions.

- Suppose that the series expansion of y_k converges on $\{|\xi| < \rho\}$.

  Binary splitting $\rightsquigarrow y(\xi)$ for $|\xi| \leqslant \frac{1}{2}\rho$ of bit size $O(1)$ in $\tilde{O}(t)$ bit ops.
- Derivatives have the same radius of convergence, are still D-finite.

  $\rightsquigarrow (y(\xi), y'(\xi), \ldots, y^{(r-1)}(\xi))$ in $\tilde{O}(t)$ ops
- We can do that for $y_1, \ldots, y_r \rightsquigarrow \begin{pmatrix} y_1(\xi) & \cdots & y_r(\xi) \\ \vdots & & \vdots \\ y_1^{(r-1)}(\xi) & \cdots & y_r^{(r-1)}(\xi) \end{pmatrix}$ in $\tilde{O}(t)$ ops
- By multiplying these matrices for steps corresponding to a decomposition

  $$\xi = 0.\underbrace{\xi_1}\underbrace{\xi_2\xi_3}\underbrace{\xi_4\xi_5\xi_6\xi_7}\underbrace{\xi_8\xi_9\xi_{10}\xi_{11}\xi_{12}\xi_{13}\xi_{14}\xi_{15}}\xi_{16}\xi_{17}\ldots$$

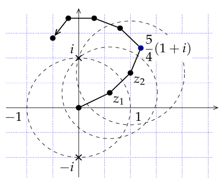
  we can evaluate the solutions at complex points of bit size t in $\tilde{O}(t)$ ops.

Fast high-precision evaluation of D-finite functions (sketch)

[Chudnovsky & Chudnovsky 1987, van der Hoeven 1999, …]

- By multiplying matrices $\begin{pmatrix} y_1(\xi) & \cdots & y_r(\xi) \\ \vdots & & \vdots \\ y_1^{(r-1)}(\xi) & \cdots & y_r^{(r-1)}(\xi) \end{pmatrix}$,

  we can also evaluate the (analytic continuation of) the solutions outside the disk $|\xi| < \rho$.



- For fixed ξ, computing $y(\xi)$ with an error $\leqslant 2^{-t}$ requires $O(t)$ digits of ξ.
- All necessary error bounds can be computed automatically.

Pseudo-theorem: "one can evaluate a **fixed** D-finite function at a **fixed** point $\in \mathbb{C}$ with an error $\leqslant 2^{-t}$ in $\tilde{O}(t)$ bit operations".

(Can be stated rigorously with more care.)

6 Rectangular splitting

Rectangular splitting for polynomials

[Paterson & Stockmeyer 1973]

Goal: evaluate $p(\xi) = a_{d-1}\,\xi^{d-1} + \cdots + a_0$ with "small" a_i at a "large" (p-bit) $\xi \in \mathbb{R}$

$$\begin{array}{rlllllll} p(x) = & (& a_0 + & a_1 x + & \cdots + & a_{\ell-1} x^{\ell-1}) & x^0 & (d = m\,\ell) \\ + & (& a_\ell + & a_{\ell+1} x + & \cdots + & a_{2\ell-1} x^{\ell-1}) & x^\ell & \\ \vdots & & & & & & & \\ + & (& a_{(m-1)\ell} + & a_{(m-1)\ell+1} x + & \cdots + & a_{m\ell-1} x^{\ell-1}) & x^{(m-1)\ell} & \end{array}$$

Algorithm.

1. (Baby steps) Compute $\xi^2, \ldots, \xi^\ell$ — $O(\ell)$ costly ops
2. Evalute the inner polynomials — $O(\ell\,m)$ cheap ops
3. (Giant steps) Compute $\xi^{2\ell}, \ldots, \xi^{(m-1)\ell}$ — $O(m)$ costly ops
4. Evaluate the outer polynomial — $O(m)$ costly ops

Same idea for evaluating $p \in \mathbb{K}[x]$ on a polynomial / matrix / …

Rectangular splitting for hypergeometric series

$$f(x) = a_0 + a_0\,a_1\,x + a_0\,a_1\,a_2\,x^2 + \cdots \qquad a_n = p(n)/q(n)$$

$$\begin{array}{rll} & & a_0\,(1 + a_1\,(x + a_2\,(x^2 + \cdots + a_{\ell-1}\,x^{\ell-1})))\;\; x^0 \\ + & a_0 \cdots a_{\ell-1} \Big(& a_\ell\,(1 + a_{\ell+1}\,(x + a_{\ell+2}\,(x^2 + \cdots + a_{2\ell-1}\,x^{\ell-1})))\;\; x^\ell \\ + & a_\ell \cdots a_{2\ell-1} \Big(& \cdots \\ + & a_{(m-1)\ell} \cdots a_{m\ell-1} \Big(& a_{(m-1)\ell}\,(1 + \cdots(\cdots(\;\cdots\; + a_{m\ell-1}\,x^{\ell-1})))\;\; x^{(m-1)\ell}\Big)\cdots\Big)\Big) \end{array}$$