Differentially Finite Power Series

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October 26, 2023
Exercises from last week
Exercise 1

The aim of this exercise is to prove algorithmically the following identity:

\[
3\sqrt[3]{\sqrt{2}} - 1 = 3\sqrt[3]{\frac{1}{9}} - 3\sqrt[3]{\frac{2}{9}} + 3\sqrt[3]{\frac{4}{9}}.
\]  

(1)

Let \( a = 3\sqrt{2} \) and \( b = 3\sqrt[3]{\frac{1}{9}} \).

a) Determine \( P_c \in \mathbb{Q}[x] \) annihilating \( c = 1 - a + a^2 \), using a resultant.

b) Deduce \( P_R \in \mathbb{Q}[x] \) annihilating the RHS of (1), using another resultant.

c) Show that the polynomial computed in b also annihilates the LHS.

d) Conclude.
Exercise 2

Let $\mathbb{K}$ be a field of characteristic zero. Consider $F \in \mathbb{K}[[x]]$ with $F(0) = 1$.

a) What is the complexity of computing $\sqrt{F}$, by using $\sqrt{F} = \exp\left(\frac{1}{2} \log F\right)$?

b) Describe a Newton iteration that directly computes $\sqrt{F}$, without appealing to successive logarithm and exponential computations.

c) Estimate the complexity of the algorithm in b.
Summary of last week’s results about Newton’s method

Let \( f \in \mathbb{K}[[x]] \).

- \( \frac{1}{f} + O(x^n) \) (when \( f(0) \neq 0 \)) in \( O(M(n)) \) ops.  
  (Newton iteration for \( 1/g - f = 0 \).)

- \( \log(f) \) (when \( f(0) = 1 \)) in \( O(M(n)) \) ops.  
  (Use \( \log(f) = \int \frac{f'}{f} \).)

- \( \exp(f) \) (when \( f(0) = 0 \)) in \( O(M(n)) \) ops.  
  (Newton iteration for \( \log(g) = f \).)

- Euclidean division \( F = Q G + R \) with \( F, G \in \mathbb{K}[x]_{<n} \) in \( O(M(n)) \) ops.  
  (Expand \( F/G \) at \( \infty \) using the Newton iteration for the inverse.)
Exercise. Consider $F \in \mathbb{K}[[x]]$ with $F(0) = 1$.

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Solution.
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a) What is the complexity of computing $\sqrt{F}$, by using $\sqrt{F} = \exp\left(\frac{1}{2} \log F\right)$?

Solution.

$L := \log(F) + O(x^n) \quad O(M(n))$

$K := L/2 \quad O(n)$

$R := \exp(K) + O(x^n) \quad O(M(n))$

Total $O(M(n))$. 
**Exercise.** Consider $F \in \mathbb{K}[[x]]$ with $F(0) = 1.$

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Total $O(M(n))$.

To be complete, we should ensure that performing each step at precision $n$ is enough:

$$\tilde{F} = F + O(x^n) \quad \Rightarrow \quad \tilde{F} = F (1 + O(x^n))$$

$$\Rightarrow \quad \log(\tilde{F}) = \log(F) + \log(1 + O(x^n)) = \log(F) + O(x^n)$$

$$\Rightarrow \quad \exp\left(\frac{1}{2} \log(\tilde{F}) + O(x^n)\right) = \sqrt{F} (1 + O(x^n)) = \sqrt{F} + O(x^n).$$
Exercise. Consider $F \in \mathbb{K}[[x]]$ with $F(0) = 1$.

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Solution. We want to solve $\Phi(R) = R^2 - F = 0$. Writing

$$\Phi(\tilde{R} + H) = (\tilde{R} + H)^2 - F = 0$$
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$$\Phi(\tilde{R} + H) = (\tilde{R} + H)^2 - F = \tilde{R}^2 - F + 2\tilde{R}H + H^2$$

leads to $H = \frac{F - \tilde{R}^2}{2\tilde{R}} - H^2$, suggesting the iteration $N(\tilde{R}) = \tilde{R} + \frac{1}{2} \left( \frac{F}{\tilde{R}} - \tilde{R} \right)$. 
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Algorithm.

$\tilde{R} := 1; \ell := 1$

while $\ell < n$

$Q = F / \tilde{R} + O(x^{2 \ell})$

$\tilde{R} := \tilde{R} + (Q - \tilde{R}) / 2$

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Let us show that, after each iteration, \( \tilde{R} = \sqrt{F} + O(x^\ell) \). This holds initially since \( F(0) = 1 \). Now assume \( \tilde{R} = \sqrt{F} + O(x^\ell) \):
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&= \frac{1}{2} \tilde{R} \left( (\sqrt{F} - \tilde{R})^2 + 2 \sqrt{F} \tilde{R} \right) + O(x^{2\ell})
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Solution. Cost: $\leq c M(2) + \cdots + c M(2^{k-1}) + c M(2^k)$ where $k = \lceil \lg n \rceil$, so $O(M(n))$. 
Differentially Finite Power Series
Definition

Let $\mathbb{K}$ be an effective field of characteristic zero.

**Definition.** A power series $f \in \mathbb{K}[[x]]$ is **differentially finite** when the vector space 
$$\text{span}_{\mathbb{K}(x)}(f, f', f'', \ldots) \subseteq \mathbb{K}((x))$$
generated by its iterated derivatives has **finite dimension** over $\mathbb{K}(x)$.

In other words: $f$ satisfies a **linear homogeneous differential equation**
$$a_r(x) f^{(r)}(x) + \cdots + a_1(x) f'(x) + a_0(x) f(x) = 0$$
with coefficients in $\mathbb{K}[x]$.

Differentially finite series are also called **D-finite** or **holonomic**.
D-Finite Functions

Authors: Manuel Kauers

Offers a comprehensive introduction into theory and techniques for D-finite functions
Contains many practical algorithms, carefully illustrated with detailed examples
Includes hundreds of original exercises with solutions for training

Part of the book series: Algorithms and Computation in Mathematics (AACIM, volume 30)

to be released Dec. 2023
**Definition.** For $U \subseteq \mathbb{C}$, a meromorphic function $f: U \to \mathbb{C}$ is called differentially finite when the vector space

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Series vs functions

**Definition.** For $\mathbb{U} \subseteq \mathbb{C}$, a meromorphic function $f: \mathbb{U} \rightarrow \mathbb{C}$ is called differentially finite when the vector space

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generated by its iterated derivatives has finite dimension over $\mathbb{K}(x)$.

- Suppose $a_r(0) \neq 0$ in the equation

$$a_r(x) f^{(r)}(x) + \cdots + a_1(x) f'(x) + a_0(x) f(x) = 0, \quad a_0, \ldots, a_r \in \mathbb{C}[x].$$

Then there exists a neighborhood $\mathbb{U} \subseteq \mathbb{C}$ of 0 such that, for any $(v_0, \ldots, v_{r-1}) \in \mathbb{C}^{r-1}$, this equation has a unique analytic solution with $f^{(i)}(0) = v_i$ for $i = 0, \ldots, r - 1$. 
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- A function that is analytic at 0 is D-finite if and only if its series expansion is D-finite.
Which of these series (functions) are D-finite?

- \( f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots \)
- \( f(x) = x^2 + 5x^3 + x^{12} \)
- \( f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots \)
- \( f(x) = \tan(x) = 1 + \frac{1}{3}x^3 + \cdots \)
- \( f(x) = \arctan(x) = 1 - \frac{1}{3}x^3 + \cdots \)
- \( f(x) = \sum_{k=0}^{\infty} k!x^k \)
- \( f(x) = \sum_{k=0}^{\infty} 2^k!x^k \)
- \( f(x) = \frac{\sin(x) + \exp(x)^2}{5\sqrt{x^7 + 1}} = 1 + 3x + 2x^2 + \frac{7}{6}x^3 + \cdots \)
Which of these series (functions) are D-finite?

- \( f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots \) \( f' - f = 0 \) ✔
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- $f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots$
  \[ f' - f = 0 \quad \checkmark \]
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  \[ f^{(13)} = 0 \quad \checkmark \]
- $f(x) = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots$
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- $f(x) = x^2 + 5x^3 + x^{12}$

- $f(x) = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots$
  $f^{(13)} = 0$  ✔

- $f(x) = \tan(x) = 1 + \frac{1}{3}x^3 + \cdots$  
  $\frac{f'(x)}{f(x)} = \frac{1}{2(1 + x)}$  ✔

- $f(x) = \arctan(x) = 1 - \frac{1}{3}x^3 + \cdots$

- $f(x) = \sum_{k=0}^{\infty} k!x^k$

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- \( f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots \) \( f' - f = 0 \) ✓
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- $f(x) = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots$ \quad $\frac{f'(x)}{f(x)} = \frac{1}{2(1 + x)}$ ✔
- $f(x) = \tan(x) = 1 + \frac{1}{3}x^3 + \cdots$  \quad poles ✗
- $f(x) = \arctan(x) = 1 - \frac{1}{3}x^3 + \cdots$  \quad $(1 + x^2)f'(x) = 1$ ✔
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- \( f(x) = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots \) \( \frac{f'(x)}{f(x)} = \frac{1}{2(1 + x)} \) ✔
- \( f(x) = \tan(x) = 1 + \frac{1}{3}x^3 + \cdots \) poles ✗
- \( f(x) = \arctan(x) = 1 - \frac{1}{3}x^3 + \cdots \) \( (1 + x^2) f'(x) = 1 \) ✔
- \( f(x) = \sum_{k=0}^{\infty} k! \, x^k \) \( x^2 f''(x) + (3x - 1) f'(x) + f(x) = 0 \ldots \) but see next slides ✔
- \( f(x) = \sum_{k=0}^{\infty} 2^k \, x^k \)
- \( f(x) = \frac{\sin(x) + \exp(x)^2}{\sqrt[5]{x^7 + 1}} = 1 + 3x + 2x^2 + \frac{7}{6}x^3 + \cdots \)
Which of these series (functions) are D-finite?

- \( f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots \) \( f' - f = 0 \) ✔
- \( f(x) = x^2 + 5x^3 + x^{12} \) \( f^{(13)} = 0 \) ✔
- \( f(x) = \sqrt{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^4 + \cdots \) \( \frac{f'(x)}{f(x)} = \frac{1}{2(1 + x)} \) ✔
- \( f(x) = \tan(x) = 1 + \frac{1}{3} x^3 + \cdots \) poles ✗
- \( f(x) = \arctan(x) = 1 - \frac{1}{3} x^3 + \cdots \) \( (1 + x^2) f'(x) = 1 \) ✔
- \( f(x) = \sum_{k=0}^{\infty} k! x^k \) \( x^2 f''(x) + (3x - 1) f'(x) + f(x) = 0 \ldots \text{but see next slides} \) ✔
- \( f(x) = \sum_{k=0}^{\infty} 2^k! x^k \) next slides ✗
- \( f(x) = \frac{\sin(x) + \exp(x)^2}{5\sqrt{x^7 + 1}} = 1 + 3x + 2x^2 + \frac{7}{6} x^3 + \cdots \)
Which of these series (functions) are D-finite?

- \( f(x) = \exp(x) = 1 + x + 2x^2 + 6x^3 + \cdots \)  \( f' - f = 0 \) ✓
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- \( f(x) = \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^4 + \cdots \)  \( \frac{f'(x)}{f(x)} = \frac{1}{2(1+x)} \) ✓
- \( f(x) = \tan(x) = 1 + \frac{1}{3}x^3 + \cdots \) poles ✗
- \( f(x) = \arctan(x) = 1 - \frac{1}{3}x^3 + \cdots \) \((1 + x^2) f'(x) = 1 \) ✓
- \( f(x) = \sum_{k=0}^{\infty} k! x^k \)  \( x^2 f''(x) + (3x - 1) f'(x) + f(x) = 0 \ldots \) but see next slides ✓
- \( f(x) = \sum_{k=0}^{\infty} 2^k! x^k \) next slides ✗
- \( f(x) = \frac{\sin(x) + \exp(x)^2}{5\sqrt{x^7 + 1}} = 1 + 3x + 2x^2 + \frac{7}{6}x^3 + \cdots \) later ✓
P-recursive sequences

**Definition.** A sequence \((u_n)_{n \in \mathbb{N}}\) is called **P-recursive** (or **P-finite**, or holonomic) if it satisfies a linear homogeneous recurrence relation

\[
b_s(n) u_{n+s} + \cdots + b_1(n) u_{n+1} + b_0(n) u_n = 0
\]

with coefficients in \(\mathbb{K}[n]\).

Equivalently: when (2) holds for sufficiently large \(n \in \mathbb{N}\).

Also, informally: when its shifts \((u_n)_{n \in \mathbb{N}}, (u_{n+1})_{n \in \mathbb{N}}, (u_{n+2})_{n \in \mathbb{N}}, \ldots\) generate a finite-dimensional vector space over \(\mathbb{K}(n)\). (But some care is needed to make sense of this definition!)

**Examples.**

\[
\begin{align*}
C_n &= \frac{1}{n+1}\binom{2n}{n} \\
F_n &= \frac{1}{\sqrt{5}} (\phi^n - \tilde{\phi}^n), \quad \phi, \tilde{\phi} = \frac{1 \pm \sqrt{5}}{2} \\
(n+1)! &= (n+1) n! \\
(n+2) C_{n+1} &= (4n + 2) C_n \\
F_{n+2} &= F_n + F_{n+1}
\end{align*}
\]
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

(\Rightarrow). Suppose \( \sum_{i=0}^{r} a_i(x) f^{(i)}(x) = 0 \). Substitute \( a_i(x) = \sum_{j=0}^{d} a_{i,j} x^j \) and \( f(x) = \sum_{n=0}^{\infty} f_n x^n \):
**Theorem.** A power series is differentially finite if and only if its coefficient sequence is P-recursive.

\[ \Rightarrow \): Suppose \( \sum_{i=0}^{r} a_i(x) f^{(i)}(x) = 0 \). Substitute \( a_i(x) = \sum_{i=0}^{d} a_{i,j} x^j \) and \( f(x) = \sum_{n=0}^{\infty} f_n x^n \):

\[
\sum_{i=0}^{r} \left( \sum_{j=0}^{d} a_{i,j} x^j \right) \left( \sum_{n=0}^{\infty} f_n n (n-1) \cdots (n-i) x^{n-i} \right) = 0,
\]
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

(⇒). Suppose $\sum_{i=0}^{r} a_i(x) f^{(i)}(x) = 0$. Substitute $a_i(x) = \sum_{i=0}^{d} a_{i,j} x^j$ and $f(x) = \sum_{n=0}^{\infty} f_n x^n$:

$$\sum_{i=0}^{r} \left( \sum_{j=0}^{d} a_{i,j} x^j \right) \left( \sum_{n=0}^{\infty} f_n n (n-1) \cdots (n-i) x^{n-i} \right) = 0,$$

$$\sum_{n=0}^{\infty} \sum_{i=0}^{r} \sum_{j=0}^{d} n (n-1) \cdots (n-i) a_{i,j} f_n x^{n-i+j} = 0,$$
**Theorem.** A power series is differentially finite if and only if its coefficient sequence is P-recursive.

\((\Rightarrow)\). Suppose \(\sum_{i=0}^{r} a_i(x) f^{(i)}(x) = 0\). Substitute \(a_i(x) = \sum_{i=0}^{d} a_{i,j} x^j\) and \(f(x) = \sum_{n=0}^{\infty} f_n x^n\):

\[
\sum_{i=0}^{r} \left( \sum_{j=0}^{d} a_{i,j} x^j \right) \left( \sum_{n=0}^{\infty} f_n n (n-1) \cdots (n-i) x^{n-i} \right) = 0,
\]

\[
\sum_{n=0}^{\infty} \sum_{i=0}^{r} \sum_{j=0}^{d} n (n-1) \cdots (n-i) a_{i,j} f_n x^{n-i+j} = 0,
\]

\[
\sum_{n'=0}^{\infty} \left( \sum_{i=0}^{r} \sum_{j=0}^{d} (n' + i - j) (n' + i - j - 1) \cdots (n' - j) a_{i,j} f_{n'+i-j} \right) x^{n'} = 0.
\]
(⇐). Suppose \( \sum_{i=0}^{s} b_i(n) u_{n+i} = 0 \) for all \( n \in \mathbb{N} \).
(\Leftarrow). Suppose \( \sum_{i=0}^{s} b_i(n) u_{n+i} = 0 \) for all \( n \in \mathbb{N} \).

Changing \( i \) to \( s-i \) and \( n \) to \( n-i \), we have \( \sum_{i=0}^{s} b_{s-i}(n-s) u_{n-i} = 0 \) for all \( n \geq s \).
Suppose $\sum_{i=0}^{s} b_i(n) u_{n+i} = 0$ for all $n \in \mathbb{N}$.

Changing $i$ to $s - i$ and $n$ to $n - i$, we have $\sum_{i=0}^{s} b_{s-i}(n-s) u_{n-i} = 0$ for all $n \geq s$.

We can multiply this relation by $n^s = n(n-1) \cdots (n-s+1)$ to get one that holds for $n \in \mathbb{N}$:

$$\forall n \in \mathbb{N}, \quad \sum_{i=0}^{s} \sum_{j=0}^{d} \beta_{i,j} n^j u_{n-i} = 0 \quad \text{where} \quad \sum_{j=0}^{d} \beta_{i,j} n^j = n^s b_{s-i}(n-s).$$  \hspace{1cm} (5)
Suppose \( \sum_{i=0}^{s} b_i(n) u_{n+i} = 0 \) for all \( n \in \mathbb{N} \).

Changing \( i \) to \( s - i \) and \( n \) to \( n - i \), we have \( \sum_{i=0}^{s} b_{s-i}(n-s) u_{n-i} = 0 \) for all \( n \geq s \).

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\]

Now observe that for any series \( f(x) = \sum_{n=0}^{\infty} f_n x^n \), one has

\[
x f(x) = \sum_{n=1}^{\infty} f_{n-1} x^n, \quad x f'(x) = \sum_{n=0}^{\infty} n f_n x^n.
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(⇐). Suppose \( \sum_{i=0}^{s} b_i(n) u_{n+i} = 0 \) for all \( n \in \mathbb{N} \).

Changing \( i \) to \( s - i \) and \( n \) to \( n - i \), we have \( \sum_{i=0}^{s} b_{s-i}(n-s) u_{n-i} = 0 \) for all \( n \geq s \).

We can multiply this relation by \( n^s = n (n - 1) \cdots (n - s + 1) \) to get one that holds for \( n \in \mathbb{N} \):

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\forall n \in \mathbb{N}, \quad \sum_{i=0}^{s} \sum_{j=0}^{d} \beta_{i,j} n^j u_{n-i} = 0 \quad \text{where} \quad \sum_{j=0}^{d} \beta_{i,j} n^j = n^s b_{s-i}(n-s). 
\]

(7)

Now observe that for any series \( f(x) = \sum_{n=0}^{\infty} f_n x^n \), one has

\[
x f(x) = \sum_{n=1}^{\infty} f_{n-1} x^n, \quad x f'(x) = \sum_{n=0}^{\infty} n f_n x^n.
\]

Thus (7) implies \( \sum_{i,j} (X \circ D)^j \circ X^i(f) = 0 \) where \( \begin{cases} [X(f)](x) = x f(x), \\ [D(f)](x) = f'(x). \end{cases} \)
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

The proof gives a conversion algorithm.

Corollary. One can compute the first $n$ terms of a D-finite series in $O(n)$ ops.
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

- The proof gives a conversion algorithm.

- Differential equation of \( \text{order} \leq r \) \( \implies \) recurrence of \( \text{order} \leq d + r \)
  
  differential equation of \( \text{degree} \leq d \) \( \implies \) recurrence of \( \text{degree} \leq r \).
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

- The proof gives a conversion algorithm.
- Differential equation of \( \text{order } \leq r \) \( \text{degree } \leq d \) \( \Rightarrow \) recurrence of \( \text{order } \leq d + r \) \( \text{degree } \leq r \).
- Also holds for \( \left\{ \text{"series" } \sum_{n \in \mathbb{Z}} u_n z^n \right\} \) \( \text{sequences } (u_n)_{n \in \mathbb{Z}} \), recurrences for all \( n \in \mathbb{Z} \).
  (No issues with the initial terms in this case.)
Theorem. A power series is differentially finite if and only if its coefficient sequence is P-recursive.

The proof gives a conversion algorithm.

Differential equation of \[ \text{order } \leq r \quad \text{degree } \leq d \] \[ \iff \] recurrence of \[ \text{order } \leq d + r \quad \text{degree } \leq r. \]

Also holds for \[ \text{"series" } \sum_{n \in \mathbb{Z}} u_n z^n \]
sequences \( (u_n)_{n \in \mathbb{Z}} \), recurrences for all \( n \in \mathbb{Z} \).

(No issues with the initial terms in this case.)

Corollary. One can compute the first \( n \) terms of a D-finite series in \( O(n) \) ops.
Differential operators as skew polynomials

Algebraic framework for working with differential operators \( f \mapsto (x \mapsto \sum_i a_i(x) f^{(i)}(x)) \)

**Definition.**

\[
\mathbb{K}(x)\langle D \rangle = \left\{ \sum_{i=0}^{r} a_i(x) D^i \mid r \in \mathbb{N}, a_i \in \mathbb{K}(x) \right\}
\]

with the usual addition of polynomials, multiplication defined by \( D \cdot x = x \cdot D + 1 \) and linearity.

Alt.: \( A/(A \langle D x - 1 \rangle A) \) where \( A = \) ring of noncommutative polynomials in \( D \) over \( \mathbb{K}(x) \).
Differential operators as skew polynomials

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Alt.: \( \mathcal{A}/(\mathcal{A} \langle Dx - 1 \rangle \mathcal{A}) \) where \( \mathcal{A} = \) ring of noncommutative polynomials in \( D \) over \( \mathcal{K}(x) \).

- \( \mathcal{K}(x) \langle D \rangle \) is a ring
- Euclidean right division: \( L = Q P + R \) with \( \text{order}(R) < \text{order}(P) \)
- Greatest common right divisor: \( L_1 = Q_1 G, \ L_2 = Q_2 G, \ G \) of max order
- Least common left multiple: \( M = U_1 L_1 = U_2 L_2 \) of min order
Recurrence operators as skew polynomials

Definition.

\[ \mathbb{K}(n)\langle S \rangle = \left\{ \sum_{i=0}^{s} b_i(n) S^i \middle| s \in \mathbb{N}, b_i \in \mathbb{K}(n) \right\} \]

with the usual addition of polynomials, multiplication defined by \( S \cdot n = (n + 1) \cdot S \) and linearity.

- Also a skew Euclidean ring
Recurrence operators as skew polynomials

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\[ \mathbb{K}(n) \langle S \rangle = \left\{ \sum_{i=0}^{s} b_i(n) S^i \mid s \in \mathbb{N}, \ b_i \in \mathbb{K}(n) \right\} \]

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- Also a skew Euclidean ring
- Diff. eq. ↔ rec. correspondence:

\[ \mathbb{K}[x, x^{-1}] \langle D \rangle \cong \mathbb{K}[n] \langle S, S^{-1} \rangle \quad \text{by} \quad \begin{cases} x \mapsto S^{-1} \\ D \mapsto (n + 1) \cdot S. \end{cases} \]
Recurrence operators as skew polynomials

Definition.

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- Diff. eq. $\leftrightarrow$ rec. correspondence:

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Exercise. Compute \( D \left( x D - 1 \right) \). Interpret the result in terms of solutions.
Equality tests

**Proposition.** Assume that \((u_n) \in \mathbb{K}^N\) and \((v_n) \in \mathbb{K}^n\) both satisfy

\[ b_s(n) y_{n+s} + \cdots + b_1(n) y_{n+1} + b_0(n) y_n = 0 \]

and \(u_n = v_n\) for \(n \leq \ell + s\) where \(\ell = \max(0, \text{largest integer root of } b_s)\). Then \(u = v\).

**Corollary.** If \(f, g \in \mathbb{K}[[x]]\) satisfy the same differential equation

\[ a_r(x) y^{(r)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = 0 \quad (a_i \in \mathbb{K}[x]) \]

one can test if \(f = g\).

Algorithm: Find the corresponding \(b_s\) and \(\ell\), test if \(f^{(n)}(0) = g^{(n)}(0)\) for \(n \leq s + \ell\).
Equality tests

**Proposition.** Assume that \((u_n) \in \mathbb{K}^N\) and \((v_n) \in \mathbb{K}^n\) both satisfy

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one can test if \(f = g\).

Algorithm: Find the corresponding \(b_s\) and \(\ell\), test if \(f^{(n)}(0) = g^{(n)}(0)\) for \(n \leq s + \ell\).

Remark: When \(a_r(0) \neq 0\), testing equality of initial conditions for \(n \leq r - 1\) suffices.
Equality tests

**Proposition.** Assume that \( (u_n) \in \mathbb{K}^N \) and \( (v_n) \in \mathbb{K}^n \) both satisfy

\[
b_s(n) y_{n+s} + \cdots + b_1(n) y_{n+1} + b_0(n) y_n = 0
\]

and \( u_n = v_n \) for \( n \leq \ell + s \) where \( \ell = \max(0, \text{largest integer root of } b_s) \). Then \( u = v \).

**Corollary.** If \( f, g \in \mathbb{K}[[x]] \) satisfy the same differential equation

\[
a_r(x) y^{(r)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = 0 \quad (a_i \in \mathbb{K}[x])
\]

one can test if \( f = g \).

\{REC, u_0, \ldots, u_{\ell+s}\} = \text{finite data structure for representing } (u_n)

\{DEQ, f(0), f'(0), \ldots, f^{(\ell+s)}\} = \text{finite data structure for representing } f
POSITIVITY CERTIFICATES FOR LINEAR RECURRENCES

ALAA IBRAHIM AND BRUNO SALVY

Abstract. We show that for solutions of linear recurrences with polynomial coefficients of Poincaré type and with a unique simple dominant eigenvalue, positivity reduces to deciding the genericity of initial conditions in a precisely defined way. We give an algorithm that produces a certificate of positivity that is a data-structure for a proof by induction. This induction works by showing that an explicitly computed cone is contracted by the iteration of the recurrence.

1. Introduction

A sequence \((u_n)_{n \in \mathbb{N}}\) of real numbers is called \(P\)-finite if it satisfies a linear recurrence

\[
p_d(n)u_{n+d} = p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n, \quad n \in \mathbb{N},
\]

with coefficients \(p_i \in \mathbb{R}[n]\) \(^{(1)}\). When the coefficients \(p_i\) are constants in \(\mathbb{R}\), the sequence is called \(C\)-finite. If \(p_d \neq 0\), the order of the relation \((1)\) is \(d\). If \(0 \notin p_d(\mathbb{N})\), then the sequence is completely determined by the recurrence and initial conditions \((u_0, \ldots, u_{d-1})\). We make this assumption in the rest of this article. \(^{(2)}\)
Proposition. If $f, g \in \mathbb{K}[[x]]$ are D-finite, one can find a differential equation

$$a_r(x) y^{(r)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = 0 \quad (a_i \in \mathbb{K}[x])$$

satisfied by both $f$ and $g$.

Corollary. If $f, g \in \mathbb{K}[[x]]$ are D-finite, then $f + g$ is D-finite.

Similarly,

- One can find a common recurrence satisfied by two given P-recursive sequences,
- The sum of two P-recursive sequences is P-recursive.
Closure by sum: direct proof

**Corollary.** If $f, g \in \mathbb{K}[[x]]$ are D-finite, then $f + g$ is D-finite.

Write $V_\varphi = \text{span}_{\mathbb{K}(x)}(\varphi^{(i)})_{i \in \mathbb{N}}$.

Since $(f + g)' = f' + g'$, one has

\[
V_{f+g} \subseteq V_f + V_g,
\]

hence

\[
\dim(V_{f+g}) \leq \dim(V_f) + \dim(V_g) < \infty.
\]
Closure by sum: direct proof

**Corollary.** If \( f, g \in \mathbb{K}[[x]] \) are D-finite, then \( f + g \) is D-finite.

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Since \((f + g)' = f' + g'\), one has

\[
V_{f+g} \subseteq V_f + V_g,
\]

hence

\[
\dim(V_{f+g}) \leq \dim(V_f) + \dim(V_g) < \infty.
\]

Showing that something is D-finite / P-recursive / (other analogous property)

\[\Leftrightarrow\]  Imprisoning its derivatives / shifts / (...) in finite dimension
Closure by sum: algorithm

Suppose

\[
\begin{align*}
    a_r(x) f^{(r)}(x) + \cdots + a_1(x) f'(x) + a_0(x) f(x) &= 0, \\
    b_s(x) g^{(s)}(x) + \cdots + b_1(x) g'(x) + b_0(x) g(x) &= 0.
\end{align*}
\]

We are looking for an equation \( c_t(x) y^{(t)}(x) + \cdots + c_0(x) y(x) = 0 \) satisfied by both \( f \) and \( g \).

By taking derivatives, we can write \( f^{(i)} = \square f + \cdots + \square f^{(r-1)} \) for any \( i \in \mathbb{N} \). Idem for \( g \).

As soon as \( t + 1 > r + s \), this system has a nonzero solution \((c_0, \ldots, c_t) \in \mathbb{K}(x)^{t+1}\).
Closure by sum: remarks

- $f + g$ satisfies a differential equation of order $\leq r + s$

- For $L_1, L_2 \in \mathbb{K}(x) \langle D \rangle$:
  $$\begin{cases} 
  L_1(f) = 0 \\
  L_2(g) = 0 
  \end{cases} \Rightarrow \quad \text{lclm}(L_1, L_2) (f + g) = 0$$

- Alternative algorithm:
  noncommutative variant of the Euclidean algorithm
Closure by product

**Proposition.**

- If \( f, g \in \mathbb{K}[[x]] \) are D-finite, then \( fg \) is D-finite.
- If \( u, v \in \mathbb{K}^N \) are P-recursive, then \( uv \) is P-recursive.

Corollary: If \( f, g \in \mathbb{K}[[x]] \) are D-finite, their **Hadamard product** \( f \odot g = \sum_{n=0}^{\infty} f_n g_n x^n \) too.
Closure by product

**Proposition.**

- If \( f, g \in \mathbb{K}[x] \) are D-finite, then \( f \circ g \) is D-finite.
- If \( u, v \in \mathbb{K}^N \) are P-recursive, then \( u \circ v \) is P-recursive.

Corollary: If \( f, g \in \mathbb{K}[x] \) are D-finite, their **Hadamard product** \( f \odot g = \sum_{n=0}^{\infty} f_n g_n x^n \) too.

**Proof.** Again by linear algebra: if \( \begin{cases} V_f \text{ is generated by } f, \ldots, f^{(r-1)}, \\ V_g \text{ is generated by } g, \ldots, g^{(s-1)}, \end{cases} \)

then \( \forall k \in \mathbb{N}, \quad (f \circ g)^{(k)} \in \text{span}_{\mathbb{K}(x)} (f^{(i)} g^{(j)})_{0 \leq i \leq r-1, \ 0 \leq j \leq s-1} \). \hfill \square

**Remark.** \( f \circ g \) satisfies a differential equation of order \( \leq rs \).
Closure by product

**Proposition.**

- If $f, g \in \mathbb{K}[[x]]$ are D-finite, then $f \circ g$ is D-finite.
- If $u, v \in \mathbb{K}^N$ are P-recursive, then $u \circ v$ is P-recursive.

Corollary: If $f, g \in \mathbb{K}[[x]]$ are D-finite, their **Hadamard product** $f \circ g = \sum_{n=0}^{\infty} f_n g_n x^n$ too.

**Proof.** Again by linear algebra: if

\[
\begin{cases}
V_f \text{ is generated by } f, \ldots, f^{(r-1)}, \\
V_g \text{ is generated by } g, \ldots, g^{(s-1)},
\end{cases}
\]

then

\[
\forall k \in \mathbb{N}, \quad (f \circ g)^{(k)} \in \text{span}_{\mathbb{K}(x)}\left(f^{(i)} g^{(j)}\right)_{0 \leq i \leq r-1, 0 \leq j \leq s-1}.
\]

**Remark.** $f \circ g$ satisfies a differential equation of order $\leq rs$.

**Exercise.** Give a better order bound in the case of $f^2$. 
Closure by product

**Proposition.**
- If $f, g \in \mathbb{K}[[x]]$ are D-finite, then $f \cdot g$ is D-finite.
- If $u, v \in \mathbb{K}^\mathbb{N}$ are P-recursive, then $u \cdot v$ is P-recursive.

Corollary: If $f, g \in \mathbb{K}[[x]]$ are D-finite, their **Hadamard product** $f \odot g = \sum_{n=0}^{\infty} f_n g_n x^n$ too.

**Proof.** Again by linear algebra: if \[
\begin{cases}
V_f \text{ is generated by } f, \ldots, f^{(r-1)}, \\
V_g \text{ is generated by } g, \ldots, g^{(s-1)},
\end{cases}
\] then \[
\forall k \in \mathbb{N}, \quad (f \cdot g)^{(k)} \in \text{span}_{\mathbb{K}(x)}(f^{(i)} g^{(j)})_{0 \leq i \leq r-1, \ 0 \leq j \leq s-1}.
\]

**Remark.** $f \cdot g$ satisfies a differential equation of order $\leq r \cdot s$.

**Exercise.** Give a better order bound in the case of $f^2$. \quad (Answer: $r \cdot (r + 1) / 2$.)
Summary

**Theorem.**

- D-finite series form an effective subalgebra of \((\mathbb{K}[x], +, \times)\).
- P-recursive sequences form an effective subalgebra of \((\mathbb{K}^\mathbb{N}, +, \times)\).

Just like we could prove that \(\sqrt[3]{3\sqrt{2} - 1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}\) by computing in \(\bar{\mathbb{Q}}\), this means that we can prove identities involving

- series like \(\exp(x)\), \(\ln(1 + x)\), \(\sqrt{1 + x}\) (and many more),
- sequences like Fibonacci’s, Catalan’s (and many more)

by computing in these rings.
Problem. Prove that \( \sin(x)^2 + \cos(x)^2 = 1 \).

Solution 1. Write \( s(x) = \sin(x) \). We have \( s'' + s = 0 \).

\[
z = s^2\]
Problem. Prove that $\sin(x)^2 + \cos(x)^2 = 1$.

Solution 1. Write $s(x) = \sin(x)$. We have $s'' + s = 0$.

\[
\begin{align*}
z & = s^2 \\
z' & = 2 s s' \\
x &
\end{align*}
\]
**Automatic proof of identities**

**Problem.** Prove that $\sin(x)^2 + \cos(x)^2 = 1$.

**Solution 1.** Write $s(x) = \sin(x)$. We have $s'' + s = 0$.

\[
\begin{align*}
  z &= s^2 \\
  z' &= 2 s s' \\
  z'' &= [2 s s']' = 2 (s')^2 + 2 s s'' = 2 (s')^2 - 2 s^2
\end{align*}
\]
Problem. Prove that $\sin(x)^2 + \cos(x)^2 = 1$.

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z''' &= [2(s')^2 - 2s^2]' = 4s's'' - 4s s' = -8s s'
\end{align*}
Automatic proof of identities

**Problem.** Prove that \( \sin(x)^2 + \cos(x)^2 = 1 \).

**Solution 1.** Write \( s(x) = \sin(x) \). We have \( s'' + s = 0 \).

\[
\begin{align*}
z &= s^2 \\
z' &= 2s s' \\
z'' &= [2s s']' = 2(s')^2 + 2ss'' = 2(s')^2 - 2s^2 \quad \times \\
z''' &= [2(s')^2 - 2s^2]' = 4s's'' - 4s's' = -8s's' \\
z'''' &= 4z' = 0
\end{align*}
\]
**Problem.** Prove that \( \sin(x)^2 + \cos(x)^2 = 1. \)

**Solution 1.** Write \( s(x) = \sin(x) \). We have \( s'' + s = 0. \)

\[
\begin{align*}
z &= s^2 \\
z' &= 2s s' \\
z'' &= 2s s' + 2s s'' = 2(s')^2 - 2s^2 \\
z''' &= 4s' s'' - 4s s' = -8s s' \\
\end{align*}
\]

Same for \( s(x) = \cos(x) \). Hence \( y(x) = \sin(x)^2 + \cos(x)^2 \) satisfies \( y''' + 4y' = 0. \)
Automatic proof of identities

**Problem.** Prove that \( \sin(x)^2 + \cos(x)^2 = 1 \).

**Solution 1.** Write \( s(x) = \sin(x) \). We have \( s'' + s = 0 \).

\[
\begin{align*}
z & = s^2 \\
z' & = 2 s s' \\
z'' & = [2 s s']' = 2 (s')^2 + 2 s s'' = 2 (s')^2 - 2 s^2 \\
z''' & = [2 (s')^2 - 2 s^2]' = 4 s' s'' - 4 s s' = -8 s s'
\end{align*}
\]

Same for \( s(x) = \cos(x) \). Hence \( y(x) = \sin(x)^2 + \cos(x)^2 \) satisfies \( y''' + 4 y' = 0 \).

Now \( f(x) = 1 \) satisfies the same equation.

The initial conditions \( y(0) = 1, \ y'(0) = 0, \ y''(0) = 0 \) agree.

Since the leading coefficient does not vanish, this implies \( y = f \).
Lazy proof of identities

Solution 2. Without even computing the equations, we know that
Lazy proof of identities

Solution 2. Without even computing the equations, we know that

- any $(s^2)^{(k)}$ belongs to $\text{span}_{\mathbb{Q}}\{s^2, s', (s')^2\},$

so $\sin(x)^2$ must satisfy an ODE of order $\leq 3$, with constant coefficients,
Lazy proof of identities

**Solution 2.** Without even computing the equations, we know that

- any \((s^2)^{(k)}\) belongs to \(\text{span}_\mathbb{Q}\{s^2, ss', (s')^2\}\),
  
  so \(\sin(x)^2\) must satisfy an ODE of order \(\leq 3\), with constant coefficients,

- \(\cos(x)^2\) must satisfy the same equation,
Lazy proof of identities

Solution 2. Without even computing the equations, we know that

- any \( (s^2)^{(k)} \) belongs to \( \text{span}_\mathbb{Q}\{s^2, s', (s')^2\} \),
  
  so \( \sin(x)^2 \) must satisfy an ODE of order \( \leq 3 \), with constant coefficients,

- \( \cos(x)^2 \) must satisfy the same equation,

- \( \sin(x)^2 + \cos(x)^2 - 1 \) must satisfy an ODE of order \( \leq 4 \).
Lazy proof of identities

Solution 2. Without even computing the equations, we know that

- any \((s^2)^{(k)}\) belongs to \(\text{span}_Q\{s^2, s', (s')^2\}\),
- so \(\sin(x)^2\) must satisfy an ODE of order \(\leq 3\), with constant coefficients,
- \(\cos(x)^2\) must satisfy the same equation,
- \(\sin(x)^2 + \cos(x)^2 - 1\) must satisfy an ODE of order \(\leq 4\).

Since this equation has constant coefficients, in particular, it is nonsingular.
So it is enough to check that \(\sin(x)^2 + \cos(x)^2 - 1 = O(x^4)\).
Minimal annihilators

Remark. We found an equation of non-minimal order!

**Definition.** The **minimal annihilator** of a D-finite function $f$ is the equation

$$f^{(r)}(x) + \cdots + a_1(x) f'(x) + a_0(x) f(x) = 0, \quad a_i \in \mathbb{K}(x)$$

of minimal order with leading coefficient $= 1$.

In other words: the generator of the left ideal $\{L | L(f) = 0\} \subseteq \mathbb{K}(x)\langle D \rangle$.

(Uniqueness e.g. by gcrd)

Idea of a minimization algorithm: given a first $L$ such that $L(f) = 0$

- Find bounds on the **degrees** of possible right divisors of $L$
- Set up linear equations for $\{L' | \text{ord}(L') < \text{ord}(L) \text{ and } L'(f) = 0\}$
MINIMIZATION OF DIFFERENTIAL EQUATIONS
AND ALGEBRAIC VALUES OF $E$-FUNCTIONS

ALIN BOSTAN, TANGUY RIVOAL, AND BRUNO SALVY

ABSTRACT. A power series being given as the solution of a linear differential equation with appropriate initial conditions, minimization consists in finding a non-trivial linear differential equation of minimal order having this power series as a solution. This problem exists in both homogeneous and inhomogeneous variants; it is distinct from, but related to, the classical problem of factorization of differential operators. Recently, minimization has found applications in Transcendental Number Theory, more specifically in the computation of non-zero algebraic points where Siegel's $E$-functions take algebraic values. We present algorithms and implementations for these questions, and discuss examples and experiments.

1. INTRODUCTION

1.1. Minimization. A linear differential equation (LDE)

$$\mathcal{L}(y(z)) := a_r(z)y^{(r)}(z) + \cdots + a_0(z)y(z) = 0$$

(1)

with polynomial coefficients $a_i(z)$ in $\mathbb{Q}[z]$ is given, together with initial conditions specifying uniquely a formal power series solution $S \in \mathbb{Q}[[z]]$, i.e. $\mathcal{L}(S(z)) = 0$. In its homoge-
Proof of identities: another example

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, \quad F_1 = 1, \quad u_n = F_{n+2} F_n - F_{n+1}^2 \]

Any homog. poly. of degree 2 in \( F_n, F_{n+1}, \ldots \) belongs to \( \text{span}_{\mathbb{K}(n)}(F_n^2, F_n F_{n+1}, F_{n+1}^2) \).

We know that the sequences \( u_n, \ldots, u_{n+3} \) must be linearly dependent over \( \mathbb{Q}(n) \).

\[
\begin{align*}
  u_n &= F_{n+2} F_n - F_{n+1}^2 \\
       &= F_n^2 + F_n F_{n+1} - F_{n+1}^2 \\
  u_{n+1} &= F_{n+1}^2 + F_n F_{n+1} - F_{n+2} \\
           &= F_{n+1}^2 + F_{n+1} (F_n + F_{n+1}) - (F_n + F_{n+1})^2 \\
           &= -F_n^2 - F_n F_{n+1} + F_{n+1}^2
\end{align*}
\]

It turns out that \( u_{n+1} = -u_n \).

Since \( u_0 = F_0^2 + F_0 F_1 - F_1^2 = -1 \), we conclude that \( F_{n+2} F_n - F_{n+1}^2 = (-1)^{n+1} \) for all \( n \).
An exercise for next week

Prove the following identity of formal power series:

\[
\arcsin(x)^2 = \sum_{k=0}^{\infty} \frac{k!}{\frac{1}{2} \cdot \frac{3}{2} \cdots (k + \frac{1}{2})} \frac{x^{2k+2}}{2k + 2}.
\]

For this:

1. Check that \( y(x) = \arcsin(x) \) is solution to \( (1 - x^2) y''(x) = x y'(x) \).
2. Deduce a linear differential equation satisfied by \( z(x) = y(x)^2 \).
3. Deduce a linear recurrence relation satisfied by the coefficients of the series.
**Algebraic series**

**Definition.** A series $f \in \mathbb{K}[[x]]$ is called **algebraic** if there exists $P \in \mathbb{K}[x, y] \setminus \{0\}$ such that

$$P(x, f(x)) = 0.$$  

**Examples:** rational series, $\sqrt[3]{1 + x}, \ldots$

**Example:** generating series of non-ambiguous context-free languages are algebraic.

**Theorem.** Algebraic series are D-finite.  
[Abel 1827, Cockle 1860, Harley 1862]

More generally:
If $f \in \mathbb{K}[[x]]$ is D-finite and $g \in x \mathbb{K}[x]$ is algebraic, then $f \circ g$ is D-finite.  
(similar proof)
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have

$$\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).$$
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have

$$\left. \frac{\partial}{\partial x} P(x, f(x)) \right|_{0} = P_x(x, f(x)) + P_y(x, f(x)) f'(x).$$

Hence

$$f'(x) = -\frac{P_x(x, f(x))}{P_y(x, f(x))}.$$
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have

$$\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).$$

Hence

$$f'(x) = -\frac{P_x(x, f(x))}{P_y(x, f(x))}$$

$$Q = \frac{1}{P_y} \text{rem } P$$
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have $\frac{\partial}{\partial x} (P(x, f(x))) = P_x(x, f(x)) + P_y(x, f(x)) f'(x)$. We have $\frac{\partial}{\partial x} (P(x, f(x))) = 0$

Hence $f'(x) = -\frac{P_x(x, f(x))}{P_y(x, f(x))}$

$= -Q(x, f(x)) P_x(x, f(x))$ \quad where \quad $Q = \frac{1}{P_y} \text{rem } P$
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have

$$\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).$$

Hence

$$f'(x) = \frac{-P_x(x, f(x))}{P_y(x, f(x))}$$

$$= -Q(x, f(x)) P_x(x, f(x)) \quad \text{where} \quad Q = \frac{1}{P_y} \rem P$$

$$= R(x, f(x)) \quad R \in \mathbb{K}(x)[y].$$
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have
\[
\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).
\]

Hence
\[
f'(x) = \frac{-P_x(x, f(x))}{P_y(x, f(x))} = -Q(x, f(x)) P_x(x, f(x))
\]
where
\[
Q = \frac{1}{P_y} \text{rem } P
\]
and
\[
R \in \mathbb{K}(x)[y].
\]

Then
\[
f''(x) =
\]
Algebraic series are D-finite: proof

Wlog, suppose \( P(x, f(x)) = 0 \) with \( P \in \mathbb{K}(x)[y] \) irreducible of degree \( d \).

We have

\[
\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).
\]

Hence

\[
f'(x) = \frac{-P_x(x, f(x))}{P_y(x, f(x))}
\]

\[
= -Q(x, f(x)) P_x(x, f(x))
\]

\[
= R(x, f(x))
\]

where \( Q = \frac{1}{P_y} \) \( \text{rem} \) \( P \)

\( R \in \mathbb{K}(x)[y] \).

Then

\[
f''(x) = R_x(x, f(x)) + R_y(x, f(x)) f'(x)
\]
Algebraic series are D-finite: proof

Wlog, suppose $P(x, f(x)) = 0$ with $P \in \mathbb{K}(x)[y]$ irreducible of degree $d$.

We have

$$\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).$$

Hence

$$f'(x) = -\frac{P_x(x, f(x))}{P_y(x, f(x))}$$

$$= -Q(x, f(x)) P_x(x, f(x))$$

$$= R(x, f(x))$$

where

$$Q = \frac{1}{P_y} \text{rem } P$$

$$R \in \mathbb{K}(x)[y].$$

Then

$$f''(x) = R_x(x, f(x)) + R_y(x, f(x)) f'(x)$$

$$= \text{poly}(x, f(x)),$$

and so on by induction.
Algebraic series are D-finite: proof

Wlog, suppose \( P(x, f(x)) = 0 \) with \( P \in \mathbb{K}(x)[y] \) irreducible of degree \( d \).

We have

\[
\frac{\partial}{\partial x} \left( P(x, f(x)) \right) = P_x(x, f(x)) + P_y(x, f(x)) f'(x).
\]

Hence

\[
f'(x) = -\frac{P_x(x, f(x))}{P_y(x, f(x))}
\]

\[
= -Q(x, f(x)) P_x(x, f(x)) \quad \text{where} \quad Q = \frac{1}{P_y} \text{rem } P.
\]

\[
R(x, f(x))
\]

Then

\[
f''(x) = R_x(x, f(x)) + R_y(x, f(x)) f'(x)
\]

\[
= \text{poly}(x, f(x)), \quad \text{and so on by induction.}
\]

Since \( P(x, y(x)) = 0 \) any \( \text{poly}(x, f(x)) \) belongs to \( \text{span}_{\mathbb{K}(x)}\{1, f, f^2, \ldots, f^{d-1}\} \).

So \( \dim_{\mathbb{K}(x)}(f, f', f'', \ldots) \leq d \).
A riddle

What is the next term in this sequence?

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, …
A riddle

What is the next term in this sequence?

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, ...

Is it generated by a “small” differential equation / recurrence?
A riddle

What is the next term in this sequence?

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, ...

Is it generated by a “small” differential equation / recurrence?

```
sage: from ore_algebra import OreAlgebra, guess
sage: guess([1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798,
       ....:     15511, 41835, 113634, 310572, 853467],
       ....:     OreAlgebra(PolynomialRing(ZZ, 'n'), 'Sn'))
(-n - 4)*Sn^2 + (2*n + 5)*Sn + 3*n + 3
```
A riddle

What is the next term in this sequence?

1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, ...

Is it generated by a “small” differential equation / recurrence?

```python
sage: from ore_algebra import OreAlgebra, guess
sage: guess([1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798,
         ....: 15511, 41835, 113634, 310572, 853467],
         ....: OreAlgebra(PolynomialRing(ZZ, 'n'), 'Sn'))
(-n - 4)*Sn^2 + (2*n + 5)*Sn + 3*n + 3
```

..., 2356779, 6536382, 18199284, 50852019, 142547559, 400763223, 1129760415, ...

(Motzkin numbers)
Hermite-Padé approximation problem.

Given $k$ power series $f_1, \ldots, f_k \in K[[x]]$, $k$ degree bounds $d_1, \ldots, d_k$, an approximation order $\sigma$, find polynomials $p_1, \ldots, p_k \in K[x]$ such that $\deg p_i < d_i$ and

$$p_1(x) f_1(x) + \cdots + p_k(x) f_k(x) = O(x^\sigma).$$

This is a linear algebra problem!

When $\sigma$ is chosen “just right” ($\sigma = d_1 + \cdots + d_k - 1$), the tuple $(p_1, \ldots, p_k)$ is called a **Hermite-Padé approximant** of type $(d_1 - 1, \ldots, d_k - 1)$ of $f$.

**Naïve algorithm:** $O(\sigma^\theta)$ ops  
**Fast algorithm:** $O(k^\theta M(\sigma) \log \sigma)$, lectures 10–11  
[Beckermann-Labahn 1994]
Guessing annihilating equations

Suppose that we are given the first few terms $f_0, \ldots, f_{n-1}$ of a sequence.

- To guess a differential equation for the generating series $\sum_i f_i x^i$, compute Hermite-Padé approximants $(a_0, \ldots, a_r)$ of $(f, f', \ldots, f^{(r)})$ for various $(r, d)$
- Candidate operator: $L = a_0(x) + \cdots + a_r(x) D^r$
- If some terms $f_{n-1}, f_{n-2}, \ldots$ not used to obtain $L$ agree with a solution, it is plausible that $L(f) = 0$ (sometimes additional properties $\rightarrow$ certification)

- To guess an algebraic equation for $\sum_i f_i x^i$, compute Hermite-Padé approximants of $(f, f^2, \ldots, f^k)$
- To guess a recurrence for $(f_i)_i$, proceed as above and convert

- Extremely useful in enumerative combinatorics (more on that in lecture 16)
Rational functions and C-finite sequences

**Definition.** A sequence \((u_n)_{n \in \mathbb{N}}\) is called **C-finite** when it satisfies a linear recurrence
\[
\forall n \in \mathbb{N}, \quad c_s u_{n+s} + \cdots + c_1 u_{n+1} + c_0 u_n = 0 \quad \text{with constant } c_i.
\]

**Proposition.** A power series is **rational** if and only if its coefficient sequence is **C-finite**.

\[
\text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{general series}
\]
Rational functions and C-finite sequences

**Definition.** A sequence \((u_n)_{n \in \mathbb{N}}\) is called **C-finite** when it satisfies a linear recurrence

\[
\forall n \in \mathbb{N}, \quad c_s u_{n+s} + \cdots + c_1 u_{n+1} + c_0 u_n = 0 \quad \text{with constant } c_i.
\]

**Proposition.** A power series is **rational** if and only if its coefficient sequence is C-finite.

**Definition.** A **Padé approximant** of type \((m-1, n-1)\) of \(f \in \mathbb{K}[[x]]\) is a pair \((p, q) \in \mathbb{K}[x]^2\) with \(\deg p < m\), \(\deg q < n\), and \(p / q = f + O(x^{m+n-1})\).

- Padé: rational series (C-finite sequence) \(\rightarrow\) polynomial fraction
- Can be computed (fast) by a variant of the (fast) Euclidean algorithm
Hypergeometric sequences and series

Definition.

- A sequence \((u_n)_{n \in \mathbb{N}}\) is **hypergeometric** if it satisfies a first-order recurrence relation with polynomial coefficients.

  In other words: \(\frac{u_{n+1}}{u_n} \in \mathbb{K}(n)\) [coincides with a rat. function for large enough \(n\)].

- A **generalized hypergeometric series** is a power series whose coefficient sequence is hypergeometric. Notation:

\[
pFq\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right| x) = \sum_{n=0}^{\infty} u_n x^n \quad \text{where} \quad u_{n+1} = \frac{\prod_i (n + a_i)}{(n + 1) \prod_j (n + b_j)} u_n, \quad u_0 = 1.
\]

- \((1 - x)^a = _1F_0(-a; ; x)\), \(\ln(1 + x) = x \ _2F_1(1, 1; 2; -x)\), \(\text{Li}_2(x) = x \ _3F_2(1, 1, 1; 2, 2; x)\), etc.

- Many identities, e.g., \(\ _2F_1(2a, 2b; a+b+\frac{1}{2}; x) = \ _2F_1(a, b; a+b+\frac{1}{2}; 4x(1-x))\) (Kummer)
Classes of power series

- Rational
- Algebraic
- D-finite
- Hypergeometric
Several variables

- The idea of D-/P-finiteness generalizes to functions of several variables:
  \[(n-k)^2 \binom{n+k}{k}, \quad e^{-x^2} \sin(\alpha x), \quad \ldots\]

- Diff. equations / recurrences are replaced by systems (⇒ finitely many ini. cond.):
  \[u_{n,k} = \binom{n}{k} \quad \leftrightarrow \quad \begin{cases} (n+1-k)u_{n+1,k} = (n+1)u_{n,k} \\ (k+1)u_{n,k+1} = (n-k)u_{n,k} \end{cases}\]

- Equations can mix derivatives and shifts (and other operators):
  \[\begin{cases} x J'_n(x) + x J_{n+1}(x) - n J_n(x) = 0 \\ x J_{n+2}(x) - 2(n+1) J_{n+1}(x) + x J_n(x) = 0 \end{cases} \quad \text{(Bessel functions)}\]

- The closure properties extend (⇒ proofs of identities)
  (algorithms based on noncommutative Gröbner bases)
Creative telescoping

New closure property: by definite summation / integration (under assumptions)

\[ u_{n,k} = \binom{n}{k} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l}
(n + 1 - k) u_{n+1,k} - (n + 1) u_{n,k} = 0 \\
(k + 1) u_{n,k+1} - (n - k) u_{n,k} = 0
\end{array} \right. \]

\[ v_n = \sum_k u_{n,k} \quad \Longleftrightarrow \quad v_{n+1} - 2v_n = 0 \]

Leads to automatic proofs of many more identities:

\[ \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{k} \right) = \left( \frac{n}{2} + 1 \right) 2^{3n} - 3 n 2^{n-2} \binom{2n}{n} \]

\[ \int_{0}^{\infty} \frac{x e^{-px^2}}{x} J_n(bx) I_n(cx) \, dx = \frac{1}{2p} \exp \left( \frac{c^2 - b^2}{4p} \right) J_n \left( \frac{bc}{p} \right) \]

\[ \ldots \]

[Zeilberger 1990, Chyzak 2000, …]
## Implementations

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Another exercise for next week

Sketch two algorithms for computing the first $n$ terms of the series

$$\exp\left( x (\sqrt{1+x} + \sqrt{2+x} + \cdots + \sqrt{k+x}) \right)$$

in $\tilde{O}(n)$ ops for fixed $k$:

- one based on Newton iteration,
- one based on D-finiteness.

What are their respective complexities w.r.t. $n$?

Which do you think is better in practice? (Some handwaving is okay.)